On Generalized and Extended Generalized $\phi$-recurrent Sasakian Finsler Structures

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Abstract: In this research, generalized and extended generalized $\phi$-recurrent Sasakian Finsler structures on horizontal and vertical tangent bundles and their various geometric properties are studied.

Genelleştirilmiş ve Genişletilmiş Genelleştirilmiş $\phi$-tekrarlı Sasakian Finsler Yapılar Üzerine

Anahtar Kelimeler
Genelleştirilmiş $\phi$-tekrarlı,
Genişletilmiş genelleştirilmiş $\phi$-tekrarlı,
Sasakian Finsler yapı,
Einstein manifold

Özet: Bu araştırmada, yatay ve dikey tanjant demetleri üzerinde genelleştirilmiş ve genişletilmiş genelleştirilmiş $\phi$-tekrarlı Sasakian Finsler yapılar ve bunların çeşitli geometrik özellikleri çalışıldı.

1. Introduction

Ruse defined a Riemannian space of the recurrent curvature for which the covariant derivation of the Riemannian curvature tensor $R$ satisfies the relation:

$$\nabla R(p, q) = A(s)R(p, q)$$

at all points for the non-zero 1-form $A$, in 1949 [13]. In this relation, if $A$ vanishes so the space is reduced to a locally symmetric manifold. Besides, generalized recurrent manifolds take part in the literature with Dubey’s study in 1979 [8]. Dubey weakened the recurrence condition that defined in (1) in the following way:

$$\nabla R(p, q) = A(s)R(p, q) + B(s)g(p, q)$$

for all vector fields $p, q, r, s$ and non-zero 1-forms $A$ and $B$ satisfying:

$$A(s) = g(s, p_1)$$

where $p_1, p_2$ are vector fields associated with $A$ and $B$, respectively and the Riemannian metric tensor $g$ is defined as follows:

$$g(p, q) = g(q, p) - g(p, r)g,$$

The Riemannian space satisfying (2) (so, (3) and (4) ) is called generalized (Riemann) recurrent manifold. Additionally, generalized Ricci recurrent and generalized concircular recurrent manifolds are defined with the following relations, respectively:

$$\nabla S(p, q) = A(s)S(p, q) + B(s)g(p, q)$$

$$\nabla C(p, q) = A(s)C(p, q) + B(s)g(p, q)$$

for all vector fields $p, q, r, s$ where $S$ is Ricci curvature tensor and $C$ is concircular curvature tensor. The Sasakian manifold satisfying

$$\phi^2(\nabla R(p, q)) = 0$$

is introduced as locally $\phi$-symmetric manifold by Takashashi, in 1977 [17]. In addition, generalized $\phi$-recurrency is one type of the weakened extensions of locally $\phi$-symmetry. $\phi$-recurrency of the spaces introduced by De, Shaikh and Biswas for Sasakian manifolds in 2003 [6] in which $\phi$-recurrent Sasakian manifold satisfies the following relation:

$$\phi^2(\nabla R(p, q)) = A(s)R(p, q)$$

for all vector fields $p, q, r, s$ and if $A = 0$ it turns to a locally $\phi$-symmetric manifold. Then generalized $\phi$-recurrency of Kenmotsu manifolds are studied by Basari and Murathan [2]. Furthermore, generalized $\phi$-recurrent spaces, like Sasakian [1], P-Sasakian [15], LP-Sasakian [14], Kenmotsu [4] and trans-Sasakian [7], are discussed in many studies. In [1], the generalized $\phi$-recurrent Sasakian manifold is defined with the following relation:

$$\phi^2(\nabla R(p, q)) = A(s)R(p, q) + B(s)g(p, q)$$
for all vector fields $p, q, r, s$. By taking $R = C$ and $R = P$ in (9) the Sasakian manifold said to be generalized $C - \phi$-recurrent and generalized $P - \phi$-recurrent where $C, P$ are concircular and projective curvature tensors, respectively. Moreover, extended generalized $\phi$-recurrence is one type of the extensions of $\phi$-recurrence and discussed by Prakash [11] and Jaiswal and Yadav [9] for Sasakian and trans-Sasakian manifolds. In [11], extended generalized $\phi$-recurrent Sasakian manifold is defined in the following way:

\[
((\nabla \rho)(p, q)) r = A(s)\phi^2(R(p, q)) r + B(s)\phi^2((g(p, q)) r)
\]  

(10)

for all vector fields $p, q, r, s$. Particularly, substituting $R$ by $C$ and $P$ respectively, the Sasakian manifold is called extended generalized $C - \phi$-recurrent and extended generalized $P - \phi$-recurrent respectively.

These studies motivated us to discuss generalized $\phi$-recurrent and extended generalized $\phi$-recurrent Sasakian Finsler structures.

2. Preliminaries

Assume that $M^{n} = (2n + 1)$, $F = (M, F)$ and $g$ be a smooth manifold, a Finsler manifold and a Finsler metric tensor with $g_{ij}(x, y) = \frac{1}{2} \delta_{ij} y^{2}$ coefficients respectively. Besides, $x = (x^{1}, \ldots, x^{m})$ are the local coordinates of $M$, $T_{x}M$ is an $m$-dimensional tangent space at $x \in M$ and $y = y^{\alpha} \frac{\partial}{\partial x^{\alpha}} \in T_{x}M$. So, $TM$ denotes $2m$-dimensional slit tangent bundle of $M$ and $u = (x, y) \in TM$ [10].” Furthermore, $T_{u}TM$ is the tangent space to $TM$ at $u$ and $(\frac{\partial}{\partial x^{r}}, \frac{\partial}{\partial y^{j}})$ are the canonical frames for $T_{u}TM$. The differential map $\pi_{u} : T_{u}TM \rightarrow T_{x}M$ satisfy $x_{u} = x(\pi_{u}) = x_{u}$. Hence, the vertical subbundle $TMV$ is derived from $ker(\pi)$. The horizontal subbundle $TMH = (N^{j}_{i}(x, y))$ is a non-linear connection on $TM$ where $N^{j}_{i} = \frac{\partial y^{j}}{\partial x^{i}}$ are obtained via spray coefficients $N^{j}_{i} = \frac{1}{2} \delta_{ij} h^{k} (\frac{\partial h^{k}}{\partial x^{i}} y^{j} - \frac{\partial h^{k}}{\partial x^{j}} y^{i})$. It enables to define $p \in T_{u}TM$ with these coefficients in the following way:

\[
p = p^{i} \frac{\partial}{\partial x^{i}} + N^{j}_{i}(x, y) p^{j} + p^{j} \frac{\partial}{\partial y^{j}} = p^{i} \frac{\partial}{\partial x^{i}} + p^{j} \frac{\partial}{\partial y^{j}}
\]

Here, $(dx^{r}, dy^{j})$ are the dual frames of $(\frac{\partial}{\partial x^{r}}, \frac{\partial}{\partial y^{j}})$ where $\delta y^{j} = dy^{j} + N^{j}_{i} dx^{i}$. In this manner, $T_{u}TM = T_{u}TM \oplus T_{u}TM$ at $u \in TM$ gives rise to complementary distributions $TMH = \bigcup_{u \in TMH} T_{u}TM$ and $TMV = \bigcup_{u \in TMV} T_{u}TM$ [5]. Furthermore, distributing $\eta = \eta dx^{i} + \eta \delta y^{j} \in (T_{u}TM)^{*}$ to horizontal and vertical parts, we have $\eta^{H} \in (T_{u}TM)^{*}$ and $\eta^{V} \in (T_{u}TM)^{*}$. The Sasaki-Finsler metric $G$ on $TM$ is defined as follows:

\[G(p, q) = G^{H}(p^{H}, q^{H}) + G^{V}(p^{V}, q^{V})\]

So, some warped contact structures with Finsler coefficients can be constructed like in [12] and [16].

**Definition 2.1.** Suppose that $(\phi^{H}, \xi^{H}, \eta^{H}, G^{H})$ and $(\phi^{V}, \xi^{V}, \eta^{V}, G^{V})$ be Sasakian Finsler structures on $TMH$ and $TMV$, respectively. Then we have the below relations:

\[
\phi \phi = -I_{n} + \eta^{H} \otimes \xi^{H} + \eta^{V} \otimes \xi^{V},
\]  

(11)

\[
\phi \xi^{H} = \phi \xi^{V} = 0,
\]

(12)

\[
\eta^{H}(\xi^{H}) = \eta^{V}(\xi^{V}) = 1,
\]

(13)

\[
\eta^{H}(\phi p^{H}) = 0, \eta^{V}(\phi p^{V}) = 0,
\]

(14)

\[
G^{H}(\phi p^{H}, \phi q^{H}) = G^{H}(p^{H}, q^{H}) - \eta^{H}(p^{H}) \eta^{H}(q^{H}),
\]

\[
G^{V}(\phi p^{V}, \phi q^{V}) = G^{V}(p^{V}, q^{V}) - \eta^{V}(p^{V}) \eta^{V}(q^{V})
\]

(15)

\[
G^{H}(p^{H}, \xi^{H}) = \eta^{H}(p^{H}), G^{V}(p^{V}, \xi^{V}) = \eta^{V}(p^{V})
\]

(16)

where $p^{H}, q^{H}, H \in T_{u}TM$ and $p^{V}, q^{V}, V \in T_{u}TM, \xi$ is the Reeb vector field, $\eta$ is the 1-form, $G$ is the Finsler metric structure and the (1,1) tensor field $\phi$ denotes the endomorphism [18].

In the Sasakian Finsler manifolds $TMH$ and $TMV$ following relations hold:

\[
G^{H}(\phi p^{H}, \phi q^{H}) = -G^{H}(p^{H}, \phi q^{H}),
\]

\[
G^{V}(\phi p^{V}, \phi q^{V}) = -G^{V}(p^{V}, \phi q^{V})
\]

(17)

\[
\nabla^{H}_{\phi p^{H}} \xi^{H} = \frac{1}{4}[\eta^{H}(\eta^{H}) p^{H} - \eta^{H}(p^{H}) q^{H}],
\]

\[R(p^{H}, q^{H}) \xi^{H} = \frac{1}{4} G^{H}(p^{H}, q^{H}) p^{H} - G^{H}(p^{H}, q^{H}) q^{H},\]

(19)

\[
R(p^{V}, q^{V}) \xi^{V} = \frac{1}{4} [\eta^{V}(\eta^{V}) p^{V} - \eta^{V}(p^{V}) q^{V}]
\]

(20)

\[
R(p^{H}, \xi^{H}) q^{H} = \frac{1}{4} [\eta^{H}(\eta^{H}) p^{H} - \eta^{H}(p^{H}) q^{H}],
\]

\[R(p^{V}, \xi^{V}) q^{V} = \frac{1}{4} [\eta^{V}(\eta^{V}) p^{V} - \eta^{V}(p^{V}) q^{V}]
\]

(21)

\[
S(p^{H}, \xi^{H}) + S(p^{H}, q^{H}) = \frac{n}{2} [\eta^{H}(p^{H}) + \eta^{V}(p^{V})]
\]

(22)

\[
S(\phi p, \phi q) + S(p^{H}, q^{H}) = \frac{n}{2} [\eta^{H}(p^{H}) + \eta^{V}(p^{V})]
\]

(23)

for all vector fields and where $R$ is the Riemann curvature tensor field, $S$ is the Ricci tensor field and $\nabla$ is the Finsler connection on $TM$ [18]. However Sasakian Finsler structures can be constructed both on horizontal and vertical tangent subbundles, in this paper; it is studied for $TMH$, for briefness.
3. Generalized $\phi$-recurrent Sasakian Finsler structures on $TM_M$

**Definition 3.1.** The Sasakian Finsler structure $(\phi^H, \xi^H, \eta^H, G^H)$ on $TM_M$ is called generalized $\phi$-recurrent if the following relation holds:

$$\phi^2((\nabla^H_v R)(p^H, q^H)\xi^H) = A^H(\xi^H)(R(p^H, q^H)\xi^H) + B^H(\xi^H)G(p^H, q^H)\xi^H$$

(24)

for $p^H, q^H, s^H, t^H \in T^*M$ where $A^H$ and $B^H$ are the non-zero 1-forms defined by

$$A^H(\xi^H) = H^H(\xi^H, p^H)_1, B^H(\xi^H) = G^H(\xi^H, p^H)_2$$

(25)

and $p^H_1, q^H_2$ are vector fields associated with 1-forms $A^H$ and $B^H$, respectively and $G$ is defined as follows:

$$G(p^H, q^H) = \frac{1}{2}[G^H(q^H, s^H)p^H - G^H(p^H, s^H)q^H].$$

(26)

**Lemma 3.2.** In a generalized $\phi$-recurrent Sasakian Finsler manifold $T^*M$, the relation $A^H + B^H = 0$ is satisfied.

**Theorem 3.3.** A generalized $\phi$-recurrent Sasakian Finsler manifold $T^*M$ with the quadruple $(\phi^H, \xi^H, \eta^H, G^H)$ is of constant curvature $\frac{1}{2}$.

**Proof.** Due to the manifold is generalized $\phi$-recurrent then (24) is satisfied. Applying $\phi$ both sides of (24), and replacing $r^H$ with $\xi^H$, by using (11) we have

$$-((\nabla^H_v R)(p^H, q^H)\xi^H) + \eta^H((\nabla^H_v R)(p^H, q^H)\xi^H) = A^H(\xi^H)[-R(p^H, q^H)\xi^H + \eta^H(R(p^H, q^H)\xi^H)] + B^H(\xi^H)(G(p^H, q^H)\eta^H - \eta^H(G(p^H, q^H)\xi^H)\eta^H).$$

(27)

By the use of (19) and (26) the following relation holds:

$$\eta^H((\nabla^H_v R)(p^H, q^H)\xi^H = \eta^H((\nabla^H_v R)(p^H, q^H)\xi^H) + \frac{1}{4}[\eta^H(q^H)p^H - \eta^H(p^H)q^H][A^H(\xi^H) + B^H(\xi^H)].$$

(28)

With the help of (17) and (18), we get

$$\eta^H((\nabla^H_v R)(p^H, q^H)\xi^H = \frac{1}{8}[G^H(\xi^H, \phi^H)\eta^H(p^H)] - G^H(\xi^H, \phi^H)\eta^H(\eta^H) = \frac{1}{2}(R(p^H, q^H)\phi^H).$$

(29)

On the other hand, by applying $\eta^H$ to the (29) and using (20), it is found that $\eta^H((\nabla^H_v R)(p^H, q^H)\xi^H) = 0$ and so (29) takes the following form:

$$((\nabla^H_v R)(p^H, q^H)\xi^H = \frac{1}{4}[\eta^H(q^H)p^H - \eta^H(p^H)q^H][A^H(\xi^H) + B^H(\xi^H)].$$

(30)

By virtue of the right parts of (28) and (29),

$$R(p^H, q^H)\phi^H = \frac{1}{4}[G^H(\xi^H, \phi^H)\eta^H(p^H) - G^H(\xi^H, \phi^H)\eta^H(\eta^H)] + \frac{1}{2}[\eta^H(q^H)p^H - \eta^H(p^H)q^H][A^H(\xi^H) + B^H(\xi^H)]$$

(31)

Using Lemma 3.2, (31) takes the following form:

$$R(p^H, q^H)\phi^H = \frac{1}{4}[G^H(\xi^H, \phi^H)\eta^H(p^H) - G^H(\xi^H, \phi^H)\eta^H(\eta^H)]$$

(32)

By applying $\phi$ both sides of (32) and via (11) and (19), the below equation is satisfied:

$$R(p^H, q^H)s^H = \frac{1}{4}[G^H(q^H, s^H)p^H - G^H(p^H, s^H)q^H].$$

(33)

So, the generalized $\phi$-recurrent Sasakian Finsler structure on $TM_M$ is of the constant curvature $\frac{1}{2}$.

**Theorem 3.4.** Assume that $TM_M$ be a $(2n+1)$-dimensional generalized $\phi$-recurrent Sasakian Finsler manifold with the structure $(\phi^H, \xi^H, \eta^H, G^H)$ and $p^H_1$ be the associated vector field of $1$-form $A^H$ given in (25). Then the Eigen value of the Ricci tensor corresponding to the Eigen vector $p^H_1$ is $\frac{1}{2}$.

**Proof.** By virtue of (24) and (11), the following relation holds:

$$((\nabla^H_v R)(p^H, q^H)\phi^H = \eta^H((\nabla^H_v R)(p^H, q^H)\phi^H)\xi^H - A^H(\phi^H)(R(p^H, q^H)\phi^H) + \frac{1}{4}[B^H(\xi^H)(G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)].$$

(34)

Substituting $s^H, p^H, r^H$ cyclically in (34), three equations are found from which it follows that

$$((\nabla^H_v R)(p^H, q^H)\phi^H) + ((\nabla^H_v R)(q^H, s^H)p^H) + ((\nabla^H_v R)(s^H, p^H)q^H) = \eta^H((\nabla^H_v R)(p^H, q^H)\phi^H)\xi^H + \eta^H((\nabla^H_v R)(q^H, s^H)p^H)\xi^H + \eta^H((\nabla^H_v R)(s^H, p^H)q^H)\xi^H - A^H(\phi^H)(R(p^H, q^H)\phi^H) + \frac{1}{4}[B^H(\xi^H)(G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)].$$

(35)

Using Second Bianchi identity and applying Lemma 3.2,

$$A^H(\phi^H)(R(p^H, q^H)\phi^H) + A^H(\phi^H)(R(q^H, s^H)\phi^H) + \frac{1}{4}[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)] + \frac{1}{4}[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)] + \frac{1}{4}[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)] = 0.$$}

(36)

Setting $q^H = r^H$ in (36),

$$A^H(\phi^H)(R(p^H, q^H)\phi^H) + A^H(\phi^H)(R(q^H, s^H)\phi^H) + \frac{1}{4}[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)] + \frac{1}{4}[G(q^H, r^H)p^H - \eta^H(G(p^H, r^H)q^H)] = 0.$$}

(37)

By the use of (23) and contracting (37), below relation is satisfied:

$$S(s^H, p^H_1) = \frac{n}{2}A^H(s^H).$$}

(38)
4. Extended generalized $\varphi$-recurrent Sasakian Finsler structures on $TM_H$

**Definition 4.1.** The Sasakian Finsler manifold $TM_H$ admitting the quadruple $(\phi^H, \xi^H, \eta^H, G^H)$ is called extended generalized $\varphi$-recurrent if the following relation holds:

\[ \phi^2((\nabla^H R)(p^H, q^H)) = A^H(s^H)\phi^2(R(p^H, q^H)) + B^H(s^H) \]

for $p^H, q^H, s^H \in T^H_u TM$.

**Theorem 4.2.** An extended generalized $\varphi$-recurrent Sasakian Finsler manifold $TM_H$ is Einstein. Also the 1-forms $A^H$ and $B^H$ are related as

\[ A^H = -B^H. \]  

**Proof.** Accept that $TM_H$ be a $\varphi$-recurrent Sasakian Finsler manifold. Then using (11) and (24), we obtain

\[ \phi^2((\nabla^H R)(p^H, q^H)) = \eta^H((\nabla^H R)(p^H, q^H)) + A^H(s^H)G(R(p^H, q^H)) + B^H(s^H) \]

From the above relation it can be seen that

\[ G((\nabla^H R)(p^H, q^H)) = \eta^H((\nabla^H R)(p^H, q^H)) + A^H(s^H)G(R(p^H, q^H)) + B^H(s^H) \]

Let \{\varepsilon^i\} $i = 1, 2, \ldots, 2n + 1$, be the orthonormal basis of $T^H_u TM$. Then putting $p^H = i^H$ and taking summation over $i$, 1 $\leq i \leq 2n + 1$, we get

\[ \sum G((\nabla^H R)(\varepsilon^i, q^H)) = \sum (\eta^H((\nabla^H R)(\varepsilon^i, q^H)) + G(R(\varepsilon^i, q^H)) \]

If the relations $\eta^H((\nabla^H R)(\varepsilon^i, Y^H)Z^H)H^H = 0$ and $\sum G(R(\varepsilon^i, q^H)) = S(q^H, r^H)$ are considered we have the following

\[ (\nabla^H S)(q^H, r^H) = A^H(s^H)S(q^H, r^H) - A^H(s^H)\eta^H(R(q^H, r^H)) \]

By using (21) in (44), we have

\[ (\nabla^H S)(q^H, r^H) = A^H(s^H)S(q^H, r^H) - \frac{1}{4} A^H(s^H)G(r^H, q^H) \]

By putting $\varphi^H = \xi^H$ in (45), we have the following:

\[ (\nabla^H S)(q^H, \xi^H) = A^H(s^H)S(q^H, \xi^H) + \frac{1}{4} B^H(s^H)\eta^H(q^H) \]

References


