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Existence and regularization of the local times of a Gaussian process

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Abstract

We study an existence result in the mean square sense of the local times of a one-dimensional Gaussian process defined by an indefinite Wiener integral. For any spatial dimension, we prove that the local times of a Gaussian process, after appropriatelly renormalized, exist as white noise distributions. We also present a regularization of the local times and show a convergence result in Hida distributions space.

Keywords: Local times, b-Gaussian process, white noise analysis.

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1. Introduction

The present paper concerns the investigation of white noise analysis approach to the local times of a certain class of Gaussian processes defined by indefinite Wiener integrals. The first idea of analyzing local times using white noise approach goes back at least to the work of Watanabe [12]. A study of local times of Brownian motion using white noise approach without renormalization was briefly mentioned in [9]. White noise technique has been further applied to the problem of local times and self-intersection local times, see e.g. [1, 4, 3, 7] just to mention a few. White noise approach to self-intersection local times has been applied also to problem in physics, see for example [2] and [5].

First of all let us fix a strictly positive real number T. The space of real-valued square-integrable function with respect to the Lebesgue measure on the interval $[0,T]$ will be denoted by $L^2([0,T])$. Let $f \in L^2([0,T])$ and $B = (B_t)_{t\in[0,T]}$ be a standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is a well-known result from Itô's stochastic calculus that the stochastic process $X = (X_t)_{t\in[0,T]}$ defined by the indefinite Wiener integral $X_t := \int_0^t f(u)\, dB_u$ is a centered Gaussian process with covariance function $\mathbb{E}\,(X_s X_t) = \int_0^{s\wedge t} f(u)^2\, du,\ s,t \geq 0$. Here $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$ and $s \wedge t$ denotes the minimum between s and t. In fact, X is also an $L^2(\mathbb{P})$-continuous martingale with respect to the natural

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filtration of B. In this work we further assume that f is bounded on $[0,T]$. We call the corresponding stochastic process as *b-Gaussian process*. By choosing f to be the constant function 1, we see that the class of b-Gaussian processes contains Brownian motion as an example. Moreover, by d-dimensional b-Gaussian process we mean the random vector $\left(X^1,\ldots,X^d\right)$ where $X^1,\ldots,X^d$ are d independent copies of one-dimensional b-Gaussian process. In [7] b-Gaussian process has been studied in the context of self-intersection local times. The main object of study in the present paper will be the *local time* of a b-Gaussian process X at a point $c\in\mathbb{R}$, which is informally defined as

$$\int_0^T \delta\left(X_t-c\right)\,dt, \tag{1.1}$$

where δ denotes the Dirac-delta function at 0. The (generalized) random variable (1.1) is intended to measure the amount of time in which the sample path of a b-Gaussian process X spends at a given point $c\in\mathbb{R}$ within the time interval $[0,T]$. A priori the expression (1.1) has no mathematical meaning. One common way to give such expression a sense is via an approximation using a Dirac sequence. More precisely, we interpret (1.1) as the limiting object of the approximated local time $\mathcal{L}_{X,\varepsilon}(T)$ of b-Gaussian process X defined by $\mathcal{L}_{X,\varepsilon}(T):=\int_0^T p_\varepsilon\left(X_t-c\right)\,dt$, $\varepsilon>0$, as $\varepsilon\to 0$, where p_ε is the heat kernel $p_\varepsilon(x):=\frac{1}{\sqrt{2\pi\varepsilon}}\exp\left(-\frac{x^2}{2\varepsilon}\right)$, $x\in\mathbb{R}$. This approximation procedure makes the limiting object, which we denote by $\mathcal{L}_X(T)$, more and more singular as the dimension of the process X increases. Hence, we need to do a *renormalization*, i.e. removal of the divergent terms, to obtain a well-defined object.

Now we describe briefly our main results. First, we investigate the existence of the local times of a b-Gaussian process as the density of the occupation measure. This density does exist in dimension one and in that case we show that the local times is a well-defined object as a limit in the mean square sense. In the white noise analysis framework we investigate the local times of b-Gaussian process for any spatial dimension. Under some conditions on the dimension of the b-Gaussian process X and the number of subtracted terms in the truncated Donsker's delta function, we are able to show the existence of the (truncated) local time $\mathcal{L}_X(T)$ as a well-defined object in some white noise distribution space. We also analyze a regularization corresponding to the Gaussian approximation described above and prove a convergence result. The organization of the paper is as follows. In section 2 we summarize some of the standard facts from the theory of white noise analysis used throughout this paper. Section 3 contains a detailed exposition of the main results and their proofs. Some concluding remarks are given in the last section.

2. Elements of white noise analysis

We briefly recall some pertinent results and notions from white noise analysis. For a more comprehensive discussion we refer to [6, 11], among others. A survey on white noise analysis and its application to Feynman integral is given in [8]. Let $(\mathcal{S}'_d(\mathbb{R}),\mathcal{C},\mu)$ be the $\mathbb{R}^d$-valued white noise space, i.e., $\mathcal{S}'_d(\mathbb{R})$ is the space of $\mathbb{R}^d$-valued tempered distributions, $\mathcal{C}$ is the Borel σ-algebra generated by cylinder sets in $\mathcal{S}'_d(\mathbb{R})$, and μ is the so-called white noise measure. The probability measure μ is uniquely determined through the Bochner-Minlos theorem by fixing the characteristic function

$$C(\vec{f}):=\int_{\mathcal{S}'_d(\mathbb{R})}\exp\left(i\langle\vec{\omega},\vec{f}\rangle\right)\,d\mu(\vec{\omega})=\exp\left(-\frac{1}{2}|\vec{f}|_0^2\right)$$

for all $\mathbb{R}^d$-valued Schwartz test function $\vec{f}\in\mathcal{S}_d(\mathbb{R})$. Here $|\cdot|_0$ denotes the usual norm in the real Hilbert space $L^2_d(\mathbb{R})$ of all $\mathbb{R}^d$-valued Lebesgue square-integrable functions, and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between $\mathcal{S}'_d(\mathbb{R})$ and $\mathcal{S}_d(\mathbb{R})$. We also have the Gel'fand triple, i.e. the continuous and dense embedding $\mathcal{S}_d(\mathbb{R}) \hookrightarrow L^2_d(\mathbb{R}) \hookrightarrow \mathcal{S}'_d(\mathbb{R})$. Let f be a function in the subset of $L^2([0,T])$ consisting all real-valued bounded functions on $[0,T]$. In the frame of white noise analysis, a d-dimensional b-Gaussian process can be represented by a continuous version of the stochastic process $X = (X_t)_{t\in[0,T]}$ with $X_t := \left(\left\langle \cdot, \mathbf{1}_{[0,t]} f \right\rangle, \ldots, \left\langle \cdot, \mathbf{1}_{[0,t]} f \right\rangle\right)$, such that for independent d-tuples of Gaussian white noise $\vec{\omega} = (\omega_1, \ldots, \omega_d) \in \mathcal{S}'_d(\mathbb{R})$ it holds that $X_t(\vec{\omega}) = \left(\left\langle \omega_1, \mathbf{1}_{[0,t]} f \right\rangle, \ldots, \left\langle \omega_d, \mathbf{1}_{[0,t]} f \right\rangle\right)$, $\vec{\omega} = (\omega_1, \ldots, \omega_d) \in \mathcal{S}'_d(\mathbb{R})$. Here $\mathbf{1}_A$ denotes the indicator function of a set $A \subset \mathbb{R}$.

Let us denote the complex Hilbert space $L^2(\mathcal{S}'_d(\mathbb{R}), \mathcal{C}, \mu)$ by $L^2(\mu)$. There are several ways to construct space of white noise test functions and distributions. For example, starting from $L^2(\mu)$ and making use of the Wiener-Itô-Segal isomorphism and the second quantization operator of the Hamiltonian of a harmonic oscillator we can obtain the Gel'fand triple $(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*$. The space $(\mathcal{S})$ of white noise test functions is obtained by taking the intersection of a family of Hilbert subspaces of $L^2(\mu)$. It is equipped with the projective limit topology and has the structure of nuclear Fréchet space. The space of generalized white noise functionals $(\mathcal{S})^*$ is defined as the topological dual space of $(\mathcal{S})$. Elements of $(\mathcal{S})$ and $(\mathcal{S})^*$ are also known as *Hida test functions* and *Hida distributions*, respectively. The main example of element of $(\mathcal{S})^*$ is the d-dimensional white noise process $W_t := (\langle \cdot, \delta_t \rangle, \cdots, \langle \cdot, \delta_t \rangle)$, where δ_t is the Dirac-delta function at $t \in \mathbb{R}$. It can be considered as the (componentwise) time-derivative of the d-dimensional Brownian motion. The rest of this section is devoted to a characterization of Hida distributions. The *S-transform* of an element $\Phi \in (\mathcal{S})^*$ is defined as $(S\Phi)(\vec{f}) := \left\langle\!\left\langle \Phi, : \exp\left(\left\langle \cdot, \vec{f} \right\rangle\right) : \right\rangle\!\right\rangle$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, where $: \exp\left(\left\langle \cdot, \vec{f} \right\rangle\right) ::= C(\vec{f}) \exp\left(\left\langle \cdot, \vec{f} \right\rangle\right) \in (\mathcal{S})$, is the so-called Wick exponential and $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the dual pairing between $(\mathcal{S})^*$ and $(\mathcal{S})$. The S-transform provides a quite useful way to identify a Hida distribution $\Phi \in (\mathcal{S})^*$, in particular, when it is very hard or impossible to find the explicit form for the Wiener-Itô chaos decomposition of Φ.

2.1. Theorem. [10] *A function $F : \mathcal{S}_d(\mathbb{R}) \to \mathbb{C}$ is the S-transform of a unique Hida distribution in $(\mathcal{S})^*$ if and only if it satisfies the conditions:*

(1) *F is ray analytic, i.e., for every $\vec{f}, \vec{g} \in \mathcal{S}_d(\mathbb{R})$ the mapping $\mathbb{R} \ni \gamma \mapsto F\left(\gamma \vec{f} + \vec{g}\right)$ has an entire extension to $\gamma \in \mathbb{C}$, and*
(2) *F has growth of second order, i.e., there exist constants $K_1, K_2 > 0$ and a continuous norm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ such that $\left|F(z\vec{f})\right| \leq K_1 \exp\left(K_2 |z|^2 \left\|\vec{f}\right\|^2\right)$, for all $z \in \mathbb{C}$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$.*

There are two important consequences of the above characterization theorem. For details and proofs see [10].

2.2. Corollary. *Let $(\Omega, \mathcal{A}, \nu)$ be a measure space and $\gamma \mapsto \Phi_\gamma$ be a mapping from Ω to $(\mathcal{S})^*$. If the S-transform of Φ_γ fulfils the following two conditions:*

(1) *the mapping $\gamma \mapsto S(\Phi_\gamma)(\vec{f})$ is measurable for all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, and*
(2) *there exist $C_1(\gamma) \in L^1(\Omega, \mathcal{A}, \nu)$, $C_2(\gamma) \in L^\infty(\Omega, \mathcal{A}, \nu)$ and a continuous norm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ such that $\left|S(\Phi_\gamma)(z\vec{f})\right| \leq C_1(\gamma) \exp\left(C_2(\gamma)|z|^2 \left\|\vec{f}\right\|^2\right)$, for all $z \in \mathbb{C}$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$,*

then Φ_γ is Bochner integrable with respect to some Hilbertian norm which topologizing $(\mathcal{S})^*$. Hence $\int_\Omega \Phi_\gamma \, d\nu(\gamma) \in (\mathcal{S})^*$, and furthermore

$$S\left(\int_\Omega \Phi_\gamma \, d\nu(\gamma)\right) = \int_\Omega S(\Phi_\gamma) \, d\nu(\gamma).$$

2.3. Corollary. *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{S})^*$ such that*

(1) *for all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, $\left(S(\Phi_n)(\vec{f})\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, and*
(2) *there exist constants $K_1, K_2 > 0$ and a continuous norm $\|\cdot\|$ on $\mathcal{S}_d(\mathbb{R})$ such that $\left|S(\Phi_n)(z\vec{f})\right| \le K_1 \exp\left(K_2|z|^2 \left\|\vec{f}\right\|^2\right)$, for all $z \in \mathbb{C}$, $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, $n \in \mathbb{N}$.*

Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{S})^*$ to a unique Hida distribution $\Phi \in (\mathcal{S})^*$.

3. Local times of b-Gaussian processes

Let $f : [0, T] \to \mathbb{R}$ be a (nonrandom) Borel measurable function and λ be the Lebesgue measure in $[0, T]$. The *occupation measure* of f up to "time" T is defined by $\mu_T(A) := \lambda(\{t \in [0, T] : f(t) \in A\})$, where A is a Borel set in $\mathbb{R}$. Thus, $\mu_T(A)$ describes the amount of time spent by f in the set A during $[0, T]$. If $[0, T] \ni t \mapsto X_t(\omega) \in \mathbb{R}$ is a sample path of a stochastic process, then its occupation measure is defined in the same way, but now $\mu_T(A)$ will depend also on the sample point ω from the underlying probability space. The following theorem gives condition on which the occupation measure of a b-Gaussian process possesses a density. In that case we call the density as the local time of the b-Gaussian process.

3.1. Proposition. *Let $X = (X_t)_{t\in[0,T]}$ be a d-dimensional b-Gaussian process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $d = 1$, the occupation measure μ_T of X, i.e. $\mu_T(A) := \int_0^T \mathbf{1}_A(X_t)\,dt = \lambda(\{t \in [0, T] : X_t \in A\})$, $A \in \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$, is $\mathbb{P}$-almost surely absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$.*

Proof. A standart result from geometric measure theory states that absolute continuity of μ_T with respect to λ_d holds if for μ_T-a.e. $x \in \mathbb{R}^d$ we have

$$\liminf_{r\to 0} \frac{\mu_T(B(x,r))}{\lambda_d(B(x,r))} < \infty,$$

where $B(x, r)$ is the closed ball around x with radius r. We apply Fatou's lemma and Fubini's theorem to obtain

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d} \liminf_{r\to 0} \frac{\mu_T(B(x,r))}{\lambda_d(B(x,r))}\,d\mu_T(x)\\
&\le \frac{\Gamma(1+d/2)}{\pi^{d/2}} \liminf_{r\to 0} \frac{1}{r^d}\mathbb{E}\int_{\mathbb{R}^d} \mu_T(B(x,r))\,d\mu_T(x)\\
&= \frac{\Gamma(1+d/2)}{\pi^{d/2}} \liminf_{r\to 0} \frac{1}{r^d}\mathbb{E}\int_0^T\int_{\mathbb{R}^d} \mathbf{1}_{B(x,r)}(X_t)\,d\mu_T(x)\,dt\\
&= \frac{\Gamma(1+d/2)}{\pi^{d/2}} \liminf_{r\to 0} \frac{1}{r^d}\mathbb{E}\int_0^T\int_0^T \mathbf{1}_{B(X_t,r)}(X_s)\,ds\,dt\\
&= \frac{\Gamma(1+d/2)}{\pi^{d/2}} \liminf_{r\to 0} \frac{1}{r^d}\int_0^T\int_0^T \mathbb{P}\left(|X_t - X_s| \le r\right)\,ds\,dt\\
&\le \frac{\Gamma(1+d/2)}{\pi^{d/2}} \liminf_{r\to 0} \frac{1}{r^d}\int_0^T\int_0^T \left(\frac{1}{2\pi\int_s^t f(u)^2\,du}\right)^{d/2} \lambda_d(B(x,r))\,ds\,dt\\
&\le \frac{2}{\alpha^d(2\pi)^{d/2}}\int_0^T\int_0^t (t-s)^{-d/2}\,ds\,dt.
\end{aligned}$$

The positive constant α exists by the assumption on the function f. Notice that in the last integral we have assumed, without loss of generality, that $s < t$. Furthermore, the last integral is finite if and only if $d = 1$. □

By the Radon-Nikodym theorem, the occupation measure μ_T of X has a density function and it is a feasible measure for the time spent at a given point during the time interval $[0, T]$. Hence, it is reasonable to define local time of b-Gaussian process at a point $c \in \mathbb{R}$ during $[0, T]$ by (1.1). Now we proceed to establish the existence of (1.1) as the limiting object of a sequence of square-integrable functions.

3.2. Theorem. *The approximated local time*

$$\mathcal{L}_{X,\varepsilon}(T) := \int_0^T \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{(X_t - c)^2}{2\varepsilon}\right) dt, \quad \varepsilon > 0$$

of one-dimensional b-Gaussian process X converges in $L^2(\mathbb{P})$ as ε tends to zero.

Proof. We observe that $\mathcal{L}_{X,\varepsilon}(T) = \frac{1}{2\pi}\int_0^T \int_{\mathbb{R}} \exp(i\xi(X_t - c)) \exp\left(-\frac{\varepsilon}{2}\xi^2\right) d\xi\, dt$. Let us denote $D := \{(t_1, t_2) : 0 < t_1 < t_2 < T\}$. Hence,

$$\begin{aligned}
&\mathbb{E}\left(\mathcal{L}_{X,\varepsilon}(T)^2\right) \\
&= \mathbb{E}\left(\frac{1}{4\pi^2}\int_D \int_{\mathbb{R}^2} \exp\left(i\sum_{j=1}^2 \xi_j(X_{t_j} - c)\right) \exp\left(-\frac{\varepsilon}{2}\sum_{j=1}^2 \xi_j^2\right) d\xi\, dt\right) \\
&= \frac{1}{4\pi^2}\int_D \int_{\mathbb{R}^2} \mathbb{E}\left(\exp\left(i\sum_{j=1}^2 \xi_j(X_{t_j} - c)\right)\right) \exp\left(-\frac{\varepsilon}{2}\sum_{j=1}^2 \xi_j^2\right) d\xi\, dt \\
&= \frac{1}{4\pi^2}\int_D \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\mathrm{var}\left(\sum_{j=1}^2 \xi_j(X_{t_j} - c)\right)\right) \exp\left(-ic\sum_{j=1}^2 \xi_j\right) \\
&\quad \times \exp\left(-\frac{\varepsilon}{2}\sum_{j=1}^2 \xi_j^2\right) d\xi\, dt,
\end{aligned}$$

where $\mathrm{var}(X)$ denotes the variance of the random variable X. Note that by Lebesgue's dominated convergence theorem $\mathbb{E}\left(\mathcal{L}_{X,\varepsilon}(T)^2\right)$ converges to

$$L := \frac{1}{4\pi^2}\int_D \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\mathrm{var}\left(\sum_{j=1}^2 \xi_j(X_{t_j} - c)\right)\right) \exp\left(-ic\sum_{j=1}^2 \xi_j\right) d\xi\, dt$$

as ε tends to zero, provided

$$l := \int_D \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\mathrm{var}\left(\sum_{j=1}^2 \xi_j(X_{t_j} - c)\right)\right) d\xi\, dt < \infty.$$

We also consider

$$\begin{aligned}
&\mathbb{E}\left(\mathcal{L}_{X,\varepsilon}(T)\mathcal{L}_{X,\delta}(T)\right) \\
&= \frac{1}{4\pi^2}\int_D \int_{\mathbb{R}^2} \mathbb{E}\left(\exp\left(i\sum_{j=1}^2 \xi_j(X_{t_j} - c)\right)\right) \exp\left(-\frac{\varepsilon}{2}\xi_1^2 - \frac{\delta}{2}\xi_2^2\right) d\xi\, dt.
\end{aligned}$$

If $l < \infty$, then we also have that $\lim_{\varepsilon,\delta\to 0} \mathbb{E}\left(\mathcal{L}_{X,\varepsilon}(T)\mathcal{L}_{X,\delta}(T)\right) = L$. This implies that $\mathcal{L}_{X,\varepsilon}(T)$, $\varepsilon > 0$ is Cauchy in $L^2(\mathbb{P})$, that is $\mathbb{E}\left((\mathcal{L}_{X,\varepsilon}(T) - \mathcal{L}_{X,\delta}(T))^2\right) = \mathbb{E}\left(\mathcal{L}_{X,\varepsilon}(T)^2\right) + \mathbb{E}\left(\mathcal{L}_{X,\delta}(T)^2\right) - 2\mathbb{E}\left(\mathcal{L}_{X,\varepsilon}(T)\mathcal{L}_{X,\delta}(T)\right)$ converges to 0 as $\varepsilon, \delta \to 0$. As a consequence, $\mathcal{L}_{X,\varepsilon}(T)$ converges in $L^2(\mathbb{P})$ as ε tends to zero. Therefore, if we can show that $l < \infty$, the proof is finished. Indeed,

$$l = \int_0^T \int_0^{t_2} \int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}\mathrm{var}\left(\sum_{j=1}^2 \xi_j(X_{t_j} - c)\right)\right) d\xi\, dt_1\, dt_2$$

$$
\begin{aligned}
&= 2\pi \int_0^T \int_0^{t_2} \frac{1}{\sqrt{\mathrm{var}(X_{t_1})\mathrm{var}(X_{t_2}) - (\mathrm{cov}(X_{t_1}, X_{t_2}))^2}} \, dt_1 \, dt_2 \\
&= 2\pi \int_0^T \int_0^{t_2} \frac{1}{\sqrt{\int_0^{t_1} f(u)^2 \, du \int_{t_1}^{t_2} f(u)^2 \, du}} \, dt_1 \, dt_2 \\
&\leq \frac{2\pi}{\alpha^2} \int_0^T \int_0^{t_2} \frac{1}{\sqrt{t_1(t_2 - t_1)}} \, dt_1 \, dt_2 \\
&< \infty,
\end{aligned}
$$

where $\mathrm{cov}(X, Y)$ denotes the covariance between random variables X and Y. □

Up to this point we are able to give a meaning to the local time $\mathcal{L}_X(T)$ as a square-integrable function with respect to the probability space on which the one-dimensional b-Gaussian process is defined. Below we establish a mathematically rigorous meaning to the random variable $\mathcal{L}_X(T)$ for any d-dimensional b-Gaussian process, $d \in \mathbb{N}$. This can be done using the theory of white noise analysis. For this purpose we consider the *Donsker delta function* of b-Gaussian process which is defined as the formal composition of the Dirac-delta function $\delta_d \in \mathcal{S}'(\mathbb{R}^d)$ with a d-dimensional b-Gaussian process $(X_t)_{t\in[0,T]}$, i.e., $\delta_d\left(X_t - c\right)$, with $c \in \mathbb{R}^d$. We can give a precise meaning to the Donsker delta function as a Hida distribution.

3.3. Proposition. *Let $X = (X_t)_{t\in[0,T]}$ be a d-dimensional b-Gaussian process and $c \in \mathbb{R}^d$. The Bochner integral*

$$
\delta_d\left(X_t - c\right) := \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp\left(ix\left(X_t - c\right)\right) \, d\lambda_d(x),
$$

is a Hida distribution with S-transform $S(\delta_d\left(X_t - c\right))(\vec{f})$ given by

$$
\left(\frac{1}{2\pi \int_0^t f(u)^2 \, du}\right)^{d/2} \exp\left(-\frac{1}{2\int_0^t f(u)^2 \, du} \sum_{j=1}^d \left(\int_0^t f_j(u) f(u) \, du - c_j\right)^2\right),
$$

for all $\vec{f} = (f_1, \ldots, f_d) \in \mathcal{S}_d(\mathbb{R})$.

Proof. By direct computation we have

$$
\begin{aligned}
&S\left(\exp\left(ix\left(X_t - c\right)\right)\right)(\vec{f}) \\
&= \left\langle\!\left\langle \exp\left(ix\left(\left\langle \cdot, \mathbf{1}_{[0,t]} f\right\rangle - c\right)\right), : \exp\left(\left\langle \cdot, \vec{f}\right\rangle\right) : \right\rangle\!\right\rangle \\
&= \exp\left(-\frac{1}{2}\left|\vec{f}\right|_0^2\right) \exp\left(-ixc\right) \int_{\mathcal{S}'_d(\mathbb{R})} \exp\left(\left\langle \vec{\omega}, ix\mathbf{1}_{[0,t]} f + \vec{f}\right\rangle\right) \, d\mu(\vec{\omega}) \\
&= \exp\left(-\frac{1}{2}\left|\vec{f}\right|_0^2\right) \exp\left(-ixc\right) \exp\left(\frac{1}{2}\left|\vec{f} + ix\mathbf{1}_{[0,t]} f\right|_0^2\right) \\
&= \exp\left(-\frac{1}{2}|x|^2 \int_0^t f(u)^2 \, du\right) \exp\left(ix\left(\left\langle \vec{f}, \mathbf{1}_{[0,t]} f\right\rangle - c\right)\right),
\end{aligned}
$$

which is a measurable function of $x \in \mathbb{R}^d$ for each $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Furthermore, let $z \in \mathbb{C}$ and $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Then

$$
\begin{aligned}
&\left|S\left(\exp\left(ix\left(X_t - c\right)\right)\right)(z\vec{f})\right| \\
&\leq \exp\left(-\frac{1}{2}|x|^2 \int_0^t f(u)^2 \, du\right) \exp\left(|x||z| \left|\left\langle \vec{f}, \mathbf{1}_{[0,t]} f\right\rangle\right|\right)
\end{aligned}
$$

$$\leq \exp\left(-\frac{1}{2}|x|^2\int_0^t f(u)^2\,du\right)\exp\left(|x||z|\beta t\sum_{j=1}^{d}\sup_{u\in\mathbb{R}}|f_j(u)|\right)$$
$$\leq \exp\left(-\frac{1}{4}|x|^2\int_0^t f(u)^2\,du\right)\exp\left(\frac{\beta^2t^2}{\int_0^t f(u)^2\,du}|z|^2\|\vec{f}\|_{\infty,1}^2\right)$$
$$\leq \exp\left(-\frac{1}{4}|x|^2\alpha^2 t\right)\exp\left(\frac{\beta^2}{\alpha^2}T|z|^2\|\vec{f}\|_{\infty,1}^2\right),$$

where $\|\cdot\|_{\infty,1}$ is a continuous norm on $\mathcal{S}_d(\mathbb{R})$ defined by

$$\|\vec{f}\|_{\infty,1} := \sum_{j=1}^{d}\sup_{u\in\mathbb{R}}|f_j(u)|,$$

and for some positive constants α and β. The first factor is an integrable function of λ_d, and the second factor is constant. Hence, according to the Corollary 2.2 $\delta_d\left(X_t-c\right)\in(\mathcal{S})^*$. We may now interchange the S-transform and integration to obtain

$$\begin{aligned}
&S\left(\delta_d\left(X_t-c\right)\right)(\vec{f})\\
&=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^d} S\left(\exp\left(ix(X_t-c)\right)\right)(\vec{f})\,d\lambda_d(x)\\
&=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^d}\exp\left(-\frac{1}{2}\,|x|^2\int_0^t f(u)^2\,du\right)\\
&\quad\times\exp\left(ix\left(\left\langle\vec{f},\mathbf{1}_{[0,t]}f\right\rangle-c\right)\right)d\lambda_d(x)\\
&=\left(\frac{1}{2\pi}\right)^d\left(\frac{2\pi}{\int_0^t f(u)^2\,du}\right)^{d/2}\prod_{j=1}^{d}\exp\left(\frac{\left(i\left(\int_0^t f_j(u)f(u)\,du-c_j\right)\right)^2}{2\int_0^t f(u)^2\,du}\right)\\
&=\left(\frac{1}{2\pi\int_0^t f(u)^2\,du}\right)^{d/2}\\
&\quad\times\exp\left(-\frac{1}{2\int_0^t f(u)^2\,du}\sum_{j=1}^{d}\left(\int_0^t f_j(u)f(u)\,du-c_j\right)^2\right).
\end{aligned}$$

□

In the following, in order to simplify the notation, we denote by $p(f)$ the prefactor $\left(\frac{1}{2\pi\int_0^t f(u)^2\,du}\right)^{d/2}$. We are now in the position to prove our main results on local times $\mathcal{L}_X(T)$ and their subtracted counterparts $\mathcal{L}_X^{(N)}(T)$. First, we introduce the notion of the truncated Donsker's delta function which is well-defined due to Theorem 2.1.

3.4. Definition. The truncated Donsker delta function $\delta_d^{(N)}\left(X_t-c\right)$ is defined as the Hida distribution such that for every $\vec{f}\in\mathcal{S}_d(\mathbb{R})$ its S-transform is given by

$$\begin{aligned}
&S(\delta_d^{(N)}\left(X_t-c\right))(\vec{f})\\
&=p(f)\exp^{(N)}\left(-\frac{1}{2\int_0^t f(u)^2\,du}\sum_{j=1}^{d}\left(\int_0^t f_j(u)f(u)\,du-c_j\right)^2\right),
\end{aligned}$$

where the truncated exponential series $\exp^{(N)}$ is given by

$$\exp^{(N)}(x) := \sum_{m=N}^{\infty}\frac{x^m}{m!}.$$

3.5. Theorem. *Let $X = (X_t)_{t\in[0,T]}$ be a d-dimensional b-Gaussian process and $c \in \mathbb{R}^d$. For any pair of integers $d \geq 1$ and $N \geq 0$ such that $2N > d-2$, the Bochner integral $\mathcal{L}_X^{(N)}(T) := \int_0^T \delta_d^{(N)}(X_t - c)\, dt$ is a Hida distribution.*

Proof. From the definition of truncated Donsker's delta function we see immediately that $S\left(\delta_d^{(N)}(X_t-c)\right)(\vec{f})$ is a measurable function of t for every $\vec{f} \in \mathcal{S}_d(\mathbb{R})$. Furthermore, for every $z \in \mathbb{C}$ and $\vec{f} \in \mathcal{S}_d(\mathbb{R})$, by using Proposition 3.3 it follows that

$$
\begin{aligned}
&\left|S\left(\delta_d^{(N)}(X_t-c)\right)(z\vec{f})\right| \\
&\leq p(f)\exp^{(N)}\left(\frac{1}{2\int_0^t f(u)^2\,du}|Re(z^2)|\sum_{j=1}^d\left(\int_0^t f_j(u)f(u)\,du\right)^2\right) \\
&\quad\times\exp^{(N)}\left(\frac{1}{\int_0^t f(u)^2\,du}\sum_{j=1}^d|c_j||Re(z)|\left|\int_0^t f_j(u)f(u)\,du\right|\right) \\
&\quad\times\exp^{(N)}\left(-\frac{1}{2\int_0^t f(u)^2\,du}\sum_{j=1}^d c_j^2\right) \\
&\leq p(f)\exp^{(N)}\left(\frac{1}{2\int_0^t f(u)^2\,du}|Re(z^2)|\sum_{j=1}^d\left(\int_0^t f_j(u)f(u)\,du\right)^2\right) \\
&\quad\times\exp^{(N)}\left(\frac{1}{2\int_0^t f(u)^2\,du}Re(z)^2\sum_{j=1}^d\left(\int_0^t f_j(u)f(u)\,du\right)^2\right) \\
&\leq p(f)\exp^{(N)}\left(\frac{1}{\int_0^t f(u)^2\,du}|z|^2\sum_{j=1}^d\left(\int_0^t f_j(u)f(u)\,du\right)^2\right) \\
&\leq\left(\frac{1}{2\pi\alpha^2 t}\right)^{d/2}\exp^{(N)}\left(\frac{\beta^2}{\alpha^2}t|z|^2\left\|\vec{f}\right\|_{\infty,2}^2\right) \\
&\leq\left(\frac{1}{2\pi\alpha^2}\right)^{d/2}\left(\frac{1}{T}\right)^N t^{N-d/2}\exp\left(\frac{\beta^2 T}{\alpha^2}|z|^2\left\|\vec{f}\right\|_{\infty,2}^2\right),
\end{aligned}
$$

where $\|\vec{f}\|_{\infty,2}^2 := \sum_{j=1}^d\left(\sup_{u\in\mathbb{R}}|f_j(u)|\right)^2$ is a continuous norm on $\mathcal{S}_d(\mathbb{R})$. Note that $t^{N-d/2}$ is integrable with respect to the Lebesgue measure on $[0,T]$ if and only if $N-d/2 > -1$. Therefore we can conclude using Corollary 2.2 that $\mathcal{L}_X^{(N)}(T) \in (\mathcal{S})^*$. □

Theorem 3.5 asserts that for one-dimensional b-Gaussian process all local times $\mathcal{L}_X^{(N)}(T)$ are well-defined as Hida distributions. This fact is not really surprising since we have already known from Theorem 3.2 that $\mathcal{L}_X(T) \in L^2(\mathbb{P})$, and in particular, $\mathcal{L}_X(T) \in L^2(\mu)$. We should emphasize the result for higher dimension. For $d \geq 2$, local times only become well-defined after omission of the divergent terms which occur in the low order terms in the truncated Donsker delta function. For example, for $d = 2$ and $d = 3$ it is sufficient to take $N = 1$, which means we only need to throw away the first lower term to have $\mathcal{L}_X(T)$ as a member of $(\mathcal{S})^*$. As an immediate consequence of Theorem 3.5 we can also compute the expectation of local times $\mathcal{L}_X^{(N)}(T)$, that is $\mathbb{E}_\mu(\mathcal{L}_X^{(N)}(T)) = \int_0^T p(f)\exp^{(N)}\left(-\frac{|c|^2}{2\int_0^t f(u)^2\,du}\right)dt$. It is also clear that the expectation is finite only in dimension one, and for higher dimension ($d \geq 2$) the expectation blows up.

The renormalization procedure, i.e. dropping the divergent terms, as described in Theorem 3.5 above, motivates the study of a regularization. We define the regularized

Donsker's delta function of b-Gaussian process as

$$\delta_{d,\varepsilon}(X_t - c) := \left(\frac{1}{2\pi\varepsilon}\right)^{d/2} \exp\left(-\frac{|X_t - c|^2}{2\varepsilon}\right)$$

and the corresponding regularized local time of b-Gaussian process $\mathcal{L}^d_{X,\varepsilon}(T) := \int_0^T \delta_{d,\varepsilon}(X_t - c)\, dt$.

3.6. Theorem. *Let $X = (X_t)_{t\in[0,T]}$ be a d-dimensional b-Gaussian process and $c \in \mathbb{R}^d$. For all $\varepsilon > 0$ and $d \geq 1$ the regularized local time $\mathcal{L}^d_{X,\varepsilon}(T)$ is a Hida distribution. Moreover, if $2N > d - 2$, then the (truncated) regularized local times $\mathcal{L}^{(N)}_{X,\varepsilon}(T) := \int_0^T \delta^{(N)}_{d,\varepsilon}(X_t - c)\, dt$ converges strongly as $\varepsilon \to 0$ in $(\mathcal{S})^*$ to the (truncated) local times $\mathcal{L}^{(N)}_X(T)$.*

Proof. The first part of the proof follows again by an application of Corollary 2.2 with respect to the Lebesgue measure on $[0, T]$. For all $\vec{f} \in \mathcal{S}_d(\mathbb{R})$ we obtain

$$\begin{aligned} &S\left(\delta_{d,\varepsilon}\left(X_t - c\right)\right)(\vec{f}) \\ &= \left(\frac{1}{2\pi\left(\varepsilon + \int_0^t f(u)^2\, du\right)}\right)^{d/2} \\ &\qquad \times \exp\left(-\frac{1}{2\left(\varepsilon + \int_0^t f(u)^2\, du\right)} \sum_{j=1}^d \left(\int_s^t f_j(u) f(u)\, du - c_j\right)^2\right), \end{aligned}$$

which is evidently measurable. Hence for all $z \in \mathbb{C}$ we have

$$\begin{aligned} &\left|S\left(\delta_{d,\varepsilon}\left(X_t - c\right)\right)(z\vec{f})\right| \\ &\leq \left(\frac{1}{2\pi\left(\varepsilon + \int_0^t f(u)^2\, du\right)}\right)^{d/2} \exp\left(\frac{\beta^2}{2\left(\varepsilon + \int_0^t f(u)^2\, du\right)} t^2 |z|^2 \left\|\vec{f}\right\|^2_{\infty,2}\right). \end{aligned}$$

We observe that $\frac{t^2}{\varepsilon + \int_0^t f(u)^2\, du}$ is bounded on $[0, T]$ and $\left(\frac{1}{2\pi\left(\varepsilon + \int_0^t f(u)^2\, du\right)}\right)^{d/2}$ is integrable on $[0, T]$. By Corollary 2.2 we may then conclude that $\mathcal{L}^d_{X,\varepsilon}(T) \in (\mathcal{S})^*$, for every $\varepsilon > 0$ and $d \geq 1$. Now we have to verify the convergence of $\mathcal{L}^{(N)}_{X,\varepsilon}(T)$ as $\varepsilon \to 0$. To this end we shall use Corollary 2.3. Since for every $\vec{f} \in \mathcal{S}_d(\mathbb{R})$

$$S\left(\mathcal{L}^{(N)}_{X,\varepsilon}(T)\right)(\vec{f}) = \int_0^T S\left(\delta^{(N)}_{d,\varepsilon}\left(X_t - c\right)\right)(\vec{f})\, dt,$$

then for all $z \in \mathbb{C}$ we have, by using similar computations as in the proof of Theorem 3.5,

$$\begin{aligned} &\left|S\left(\mathcal{L}^{(N)}_{X,\varepsilon}(T)\right)(z\vec{f})\right| \\ &\leq \int_0^T \left|S\left(\delta^{(N)}_{d,\varepsilon}\left(X_t - c\right)\right)(z\vec{f})\right| dt \\ &\leq \int_0^T \left(\frac{1}{2\pi \int_0^t f(u)^2\, du}\right)^{d/2} \exp^{(N)}\left(\frac{\beta^2}{2\int_0^t f(u)^2\, du} t^2 |z|^2 \left\|\vec{f}\right\|^2_{\infty,2}\right) dt \\ &\leq \left(\frac{1}{2\pi\alpha^2}\right)^{d/2} \left(\frac{1}{T}\right)^N \left(\int_0^T t^{N-d/2}\, dt\right) \exp\left(\frac{\beta^2 T}{\alpha^2} |z|^2 \left\|\vec{f}\right\|^2_{\infty,2}\right). \end{aligned}$$

This shows the uniform boundedness condition. In particular, we have

$$\left|S\left(\delta_{d,\varepsilon}^{(N)}\left(X_t-c\right)\right)(\vec{f})\right| \leq \left(\frac{1}{2\pi\alpha^2}\right)^{d/2}\left(\frac{1}{T}\right)^{N} t^{N-d/2}\exp\left(\frac{\beta^2 T}{\alpha^2}\left\|\vec{f}\right\|_{\infty,2}^{2}\right).$$

The latter upper bound is an integrable function on $[0,T]$. Finally, Lebesgue's dominated convergence theorem and Corollary 2.3 deliver the assertion of the theorem. □

4. Conclusion

In this article, we have showed the existence of the local times of a Gaussian process X defined by an indefinite Wiener integral. In dimension one the local times of X exist as square integrable functions. In higher dimensions, by using a white noise analysis method, we proved that the renormalized local times of X are Hida distributions. Furthermore, a convergence result to the renormalized local times was also established. We observe that renormalization method (removal of divergent terms) works since in the Wiener-Itô chaos decomposition the kernel functions of increasing order are less and less singular in the L^1-sense.

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