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The growth of generalized Hadamard product of entire axially monogenic functions

M. Abdalla*† and M. Abul-Ez‡

Abstract

In this article, we estimated upper bounds for the growth order and growth type of generalized Hadamard product entire axially monogenic functions. Also, some results concerning the linear substitution are discussed. The obtained results are the natural generalizations of those given in complex setting of one variable to higher dimensions of more than four.

Keywords: Axially monogenic function, Hadamard product, Growth order, Growth type.

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1. Introduction

The study of asymptotic growth behavior of integral functions is one of the main topic in complex analysis. Such study for entire functions of one complex variable was generalized in three directions. The first direction is the study of the asymptotic growth behavior of the entire functions of several complex variables (see e.g. [12, 13, 17, 20]). The second one is devoted to the integral functions of several complex matrices in different domains, for which we may mention for examples [15]. The third direction is involved in Clifford analysis to study the asymptotic growth behavior of entire monogenic functions (for examples, [3, 5, 7, 8, 9, 10, 21]).

In [6, 11], R. Delanghe, F. Sommen and F. Brackx introduced the monogenic functions with values in a real Clifford algebra defined on a nonempty subset of R^n and

*Current address: Mathematics Department, Faculty of Science, King Khalid University, Abha, Saudi Arabia.
Permanent address: Mathematics Department, Faculty of Science, South Valley University, Qena 83523, Egypt.
Emails: moabdalla@kku.edu.sa - m.abdallah@sci.svu.edu.eg.
†Corresponding Author.
‡Mathematics Department, Faculty of Science, Sohag University, Sohag 82524, Egypt.
Email: mabulez56@hotmail.com

obtained many important monogenic function theoretic results, such as the Cauchy integral formula, the Taylor expansion, and the Laurent expansion and so on, which are the extensions of the well-known classical theorems.

Later on in [2] the notion of growth order and type was introduced for a subclass of Clifford monogenic functions which are generated by a special subclass of monogenic polynomials, denoted by special monogenic functions (or axially monogenic functions). These are functions generated by a special subfamily of monogenic polynomials. In the follow-up papers [3, 4] it was analyzed under which growth conditions the related basic set of special polynomials form a Cannon set which gave some first results on some questions around the growth of this particular subclass. Also, growth order of Hadamard product of bases for axially monogenic polynomials and its convergence properties is studied in [1]. Recently, Abul-Ez and De Almeida are investigated the growth of entire axially monogenic functions with the help of lower order and type in [5].

In this paper the mode of increase of the generalized Hadamard product of two axially monogenic functions is determined in terms of the growth order and the growth type of these axially monogenic functions. The results obtained involve the coefficients in the Taylor expansion of the generalized Hadamard product of entire axially monogenic function, obtaining their order and type. Also, linear substitution of the generalized Hadamard product of entire axially monogenic function is established.

1.1. The Clifford Toolbox. This subsection contains some definitions and basic properties of Clifford algebra, which we use throughout of this paper. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal base of the Euclidean vector space $\mathbb{R}^m$ with a product according to the multiplication rules

$$e_k e_l + e_l e_k = -2\delta_{kl}, \quad k, l = 1, \ldots, m\,,$$

where δ_{kl} denotes the Kronecker symbol. This non-commutative product generates the 2^m-dimensional Clifford algebra $Cl_{0,m}$ over $\mathbb{R}$, and the set $\{e_A : A \subseteq \{1, \ldots, m\}\}$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_r}$, $1 \leq h_1 \leq \ldots \leq h_m$, $e_\phi = e_0 = 1$, forms a basis of $Cl_{0,m}$. In this instance $Cl_{0,0}$ is the field of real numbers, $Cl_{0,1}$ the field of complex numbers and $Cl_{0,2} = \mathbb{H}$ the quaternion skew field, respectively. Canonically the real vector space $\mathbb{R}^{m+1}$ will be embedded in $Cl_{0,m}$ by identifying $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ with the element $x = x_0 + \underline{x}$ of the algebra, where $\underline{x} = e_1 x_1 + \ldots + e_m x_m$. The conjugate of x is $\overline{x} = x_0 - \underline{x}$, and the norm $|x|$ of x is defined by $|x|^2 = x\overline{x} = \overline{x}x$. As $Cl_{0,m}$ is isomorphic to $\mathbb{R}^{2^m}$ we may provide it with the $\mathbb{R}^{2^m}$-norm $|a|$, and one easily sees that for any $a, b \in Cl_{0,m}$, $|a.b| \leq 2^{\frac{m}{2}}\, |a| \,.\, |b|$, where $a = \sum_{A \subseteq M} a_A e_A$ and M stands for $\{1, 2, \ldots, m\}$.

In the sequel, we consider $Cl_{0,m}$-valued functions defined in some open subset $\Omega \subset \mathbb{R}^{m+1}$, i.e. functions of the form $f(x) = \sum_A f_A(x) e_A$, where $f_A(x)$ are real-valued functions. Suggested by the case $m = 1$, call an $Cl_{0,m}$-valued function f in $\mathbb{R}^{m+1}$ left-monogenic, which is annihilated by the generalized Cauchy-Riemann operator $D := \sum_{j=0}^{m} e_j \left(\frac{\partial}{\partial x_j}\right)$, i.e. $Df = 0$. Since the operator D can be applied either from the left- and from the right-hand side, it is usual to refer to a left- and right monogenic function, respectively. For simplicity, from now on we only deal with left monogenic functions that for simplicity we call axially monogenic. The case of right monogenic functions may be treated analogously.

2. Axially monogenic functions

In [2, 3] Abul-Ez and Constales were the first who introduced the study of the asymptotic growth behavior of axially monogenic functions, which generalizes in a natural way the analogue study for holomorphic functions of one complex variable to (m+1)-dimensional Euclidean space. Their definition of the axially monogenic functions proceeds as follows:

The right $Cl_{0,m}$-module $Cl_{0,m}[x]$, defined by

$$\mathcal{A}_m = \text{span}_{Cl_{0,m}}\{z_n(x) : n \in \mathbb{N}\}$$

is called the space of axially monogenic polynomials, if the polynomial $z_n(x)$ is given by (see [1]).

$$z_n(x) = \sum_{i+j=n} \frac{\left(\frac{m-1}{2}\right)_i \left(\frac{m+1}{2}\right)_j}{i!j!} \overline{x}^i x^j,$$

where for $b \in \mathbb{R}$, $(b)_l = b(b+1)...(b+l-1)$ and $\overline{x}$ is the conjugate of x, and $\mathbb{R}^{m+1}$ is identified with a subset of $Cl_{0,m}$. If $P_n(x)$ is a homogeneous axially monogenic polynomial of degree n in x then (see [2]) $P_n(x) = z_n(x)\alpha$, where α is some constant in $Cl_{0,m}$, and

$$(2.1) \qquad \|z_n(x)\|_r = \sup_{|x|=r} |z_n(x)| = \binom{m+n-1}{n} r^n = \frac{(m)_n}{n!} r^n,$$

where $\frac{(m)_n}{n!} = \frac{(m+n-1)!}{n!(m-1)!}$.

2.1. Definition. (**Axially monogenic function**). Let Ω be a connected open subset of $\mathbb{R}^{m+1}$ containing 0, then a monogenic function in Ω is said to be axially monogenic in Ω iff its Taylor series near zero (which is known to exist) has the form

$$(2.2) \qquad f(x) = \sum_{n=0}^{\infty} z_n(x) c_n, \;\; c_n \in Cl_{0,m}.$$

A function f is said to be axially monogenic function (or special monogenic function) on the closed ball $\overline{B}(r)$ if it is axially monogenic on some connected open neighborhood Ω_f of $\overline{B}(r)$.

The fundamental references for axially monogenic functions see for instance [16, 19].

2.2. Definition. The radius of regularity R_f of axially monogenic function is defined by

$$R_f = \frac{1}{\limsup_{n\to\infty} \|c\|^{\frac{1}{n}}}.$$

Then the axially monogenic function (2.2) is entire if $R_f = \infty$.

2.3. Proposition. ***(Cauchy's inequality)***
Let $f(x) = \sum_0^\infty z_n(x)c_n$ be an axially monogenic function defined on a neighborhood of the closed ball $\overline{B}(0,r)$. Then (see [2, 3] *).*

$$\|c_n\| \le \sqrt{\frac{n!}{(m)_n}} \left(\frac{M(r,f)}{r^n}\right),$$

where $M(r,f) = \sup_{\|x\|=r} \|f(x)\|$ is the maximum modulus of f.

In [2, 3] the following main result was given:

2.4. Theorem. *Suppose that f be an entire axially monogenic function. Then the order ρ of f is given by*

$$\rho = \limsup_{n\to\infty} \frac{(n\log n)}{\log(\|c_n\|^{-1})} \tag{2.3}$$

and if $0<\rho<\infty$, then its type τ is given by

$$\tau = \frac{1}{e\rho}\limsup_{n\to\infty} n\,\|c_n\|^{\rho/n}. \tag{2.4}$$

3. Statement of results

Relying on [3], suppose that

$$\mathcal{G}_s(x) = \sum_{n=0}^{\infty} z_n(x) a_{n,s}, \quad s=1,2$$

are two entire axially monogenic functions of the growth orders ρ_s and the growth types τ_s, then

$$\rho_s = \limsup_{n\to\infty} \frac{(n\log n)}{\log(\|a_{n,s}\|^{-1})} = \limsup_{r\to\infty} \frac{\log\log M_s(r)}{\log r} \tag{3.1}$$

If $0<\rho<\infty$, then

$$\tau_s = \frac{1}{e\rho_s}\limsup_{n\to\infty} n\,\|a_{n,s}\|^{\rho_s/n} = \limsup_{r\to\infty} \frac{\log M_s(r)}{r^{\rho_s}}, \tag{3.2}$$

where $M_s(r) = \max_{\|x\|=r} \|\mathcal{G}_s(x)\|,\ s=1,2.$

In analogy with the complex setting [14, 18], we define the generalized Hadamard product of two axially monogenic functions denoted $\mathcal{H}(x)$ as follows

$$\mathcal{H}(x) = (\mathcal{G}_1 \triangle \mathcal{G}_2)(x;\alpha,\beta) = \sum_{n=0}^{\infty} z_n(x) c_n, \tag{3.3}$$

where $c_n = a^{\alpha}_{n,1} a^{\beta}_{n,2}$ and α,β are any real numbers and $a^{\alpha}_{n,1}, a^{\beta}_{n,2}$ are the α^{th} and β^{th} powers of $a^{\alpha}_{n,1}$ and $a^{\beta}_{n,2}$ respectively.

In the special case, if we take $\alpha=\beta=1$ we get the Hadamard product $(\mathcal{G}_1 \triangle \mathcal{G}_2)(x;1,1) = (\mathcal{G}_1 \star \mathcal{G}_2)(x)$.

Next, we present an estimation of the growth order ρ and the growth type τ of the generalized Hadamard product of entire axially monogenic functions by the following theorems:

3.1. Theorem. *Suppose that $\mathcal{G}_s(x)$; $s=1,2$ are two entire axially monogenic functions of positive and finite growth orders ρ_s, $s=1,2$ Then the growth order ρ $(0<\rho<\infty)$ of the generalized Hadamard product function $\mathcal{H}(x)$ is characterized by the following inequality*

$$\rho \le \frac{\rho_1\rho_2}{\alpha\,\rho_2 + \beta\rho_1}. \tag{3.4}$$

3.2. Theorem. *Let $\mathcal{H}(x)$ be the generalized Hadamard product of entire axially monogenic functions whose its constituents are the two axially monogenic functions $\mathcal{G}_1(x)$ and*

$\mathcal{G}_2(x)$, of respective growth orders ρ_1 and ρ_2. If ρ is the growth order of $\mathcal{H}(x)$ such that $\rho = \frac{\rho_1\rho_2}{\beta\rho_1+\alpha\rho_2}$ then the growth type τ of $\mathcal{H}(x)$ is estimated by the following inequality

$$\tau \leq \frac{1}{\rho}(\rho_1\tau_1)^{\alpha(\rho/\rho_1)}\,(\rho_2\tau_2)^{\beta(\rho/\rho_2)}, \tag{3.5}$$

where τ_1 and τ_2 are the growth types of $\mathcal{G}_1(x)$ and $\mathcal{G}_2(x)$, respectively.

Proof of Theorem 3.1. Since $\mathcal{G}_s(x),\ s = 1,2$ are two entire axially monogenic functions then relying to [2, 3] we have

$$\limsup_{n\to\infty} \|a_{n,s}\|^{\frac{1}{n}} = 0, \quad s = 1,2.$$

Also, since $c_n \in Cl_{0,m}$ which gives $|c_n| \leq\ 2^{\frac{n}{2}}\ |a_{n,1}||a_{n,2}|$, thus

$$\limsup_{n\to\infty} \|c_n\|^{\frac{1}{n}} \leq\ 2^{\frac{1}{2}}\ \limsup_{n\to\infty} \|a_{n,1}\|^{\frac{1}{n}}\ \limsup_{n\to\infty} \|a_{n,2}\|^{\frac{1}{n}}.$$

Hence $\mathcal{H}(x)$ is an entire axially monogenic function.

Owing to (3.1) we have for the two orders ρ_1 and ρ_2 of the respective entire axially monogenic functions $\mathcal{G}_1(x)$ and $\mathcal{G}_2(x)$ that

$$\frac{n\log n}{-\log(\|a_{n,s}\|)} = \rho_s, \quad s = 1,2.$$

Therefore, for an arbitrary $\varepsilon > 0$ and constants N_1 and N_2, we get

$$-\log(\|a_{n,1}\|) > (\frac{1}{\rho_1} - \frac{1}{\varepsilon})n\log n, \ \text{ for } \ n > N_1$$

and

$$-\log(\|a_{n,2}\|) > (\frac{1}{\rho_2} - \frac{1}{\varepsilon})n\log n, \ \text{ for } \ n > N_2.$$

Let $n > N > \max{(N_1, N_2)}$, we obtain

$$-\log(\|a_{n,1}\|\|a_{n,2}\|) > (\frac{1}{\rho_1} + \frac{1}{\rho_2} - \varepsilon\ )n\log n \ \text{ for } \ n > N_2.$$

or,

$$\liminf_{n\to\infty} \frac{-\log(\|a_{n,1}\|\|a_{n,2}\|)}{n\log n} >\ \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Thus, we can be reduced to

$$\liminf_{n\to\infty} \frac{-\log(\|a^{\alpha}_{n,1}\|\|a^{\beta}_{n,2}\|)}{n\log n} >\ \frac{\alpha}{\rho_1} + \frac{\beta}{\rho_2}.$$

Since

$$||c_n|| \leq\ 2^{\frac{n}{2}}\ ||a^{\alpha}_{n,1}||||a^{\beta}_{n,2}||,$$

then

$$-\log||c_n|| \geq -n\log\sqrt{2} - \log(\ ||a^{\alpha}_{n,1}||||a^{\beta}_{n,2}||).$$

From which we have

$$\limsup_{n\to\infty} \frac{-\log||c_n||}{n\log n} \geq \liminf_{n\to\infty} \frac{-n\log\sqrt{2} - \log(\ ||a^{\alpha}_{n,1}||||a^{\beta}_{n,2}||)}{n\log n} \geq\ \frac{\alpha}{\rho_1} + \frac{\beta}{\rho_2}.$$

This immediately given

$$\rho \leq \frac{\rho_1\rho_2}{\beta\rho_1 + \alpha\rho_2}.$$

Thus the proof of the Theorem 3.1 is complete. □

Proof of Theorem 3.2. Since $\rho_s;\ \ s=1,2$ are positive and finite, then for finite numbers γ_s and finite positive integers N_s and using (3.2), we have

$$\|a_{n,s}\| < (\frac{e\rho_s\gamma_s}{n})^{n/\rho_s},\ \ n>N_s,\ s=1,2.$$

Letτ be the growth type of the generalized Hadamard product of entire axially monogenic functions $\mathcal{H}(x)$, then it follows from the definition of the growth type that

$$\begin{aligned}\tau =&\frac{1}{e\rho}\ \limsup_{n\to\infty} n\|c_n\|^{\rho/n}\\ \le&\frac{1}{e\rho}\ \limsup_{n\to\infty} n\left((\frac{e\rho_1\gamma_1}{n})^{\alpha n/\rho_1}\ (\frac{e\rho_2\gamma_2}{n})^{\beta n/\rho_2}\right)^{\rho/n}\\ =&\frac{1}{\rho}\Big[(\rho_1\gamma_1)^{\alpha/\rho_1}\ (\rho_2\gamma_2)^{\beta/\rho_2}\Big]^{\rho}.\end{aligned}$$

Since $\gamma_s,\ s=1,2$ can be chosen as near to τ_s as possible we infer that

$$\tau \le \frac{1}{\rho}\Big[(\rho_1\tau_1)^{\alpha/\rho_1}\ (\rho_2\tau_2)^{\beta/\rho_2}\Big]^{\rho},$$

yields the assertion (3.5) of Theorem 3.2. □

The above two theorems allow us to obtain directly the precise growth order and growth type of each the generalized Hadamard product of entire axially monogenic functions when knowing its Taylor coefficients explicitly without determining the exact value of $M(r)$ which is usually difficult to fined and even impossible in most of the cases.

The following example shows that the upper bounds of (3.4) and (3.5) are attainable.

3.3. Example. Suppose that $\mathcal{G}(x)=\sum_n^{+\infty} z_{2n}(x)\frac{1}{n!}$ and $\mathcal{G}(x)=\sum_n^{+\infty} z_{2n}(x)\frac{3^n}{n!}$ are two entire axially monogenic functions of common order 2 and of respective type 1, 3. Consider the generalized Hadamard product of entire axially monogenic function

$$\mathcal{H}(x)=\sum_n^{+\infty} z_{2n}(x)(\frac{1}{n!})^5\ (\frac{3^n}{n!})^4,$$

then the Taylor coefficient of $\mathcal{H}(x)$ is $c_{2n}=(\frac{1}{n!})^9\ 3^{4n}$.

Evaluating the growth order and growth type of $\mathcal{H}(x)$, we get

$$\begin{aligned}\rho =&\limsup_{n\to\infty}\frac{2n\log 2n}{\log(\|c_n\|^{-1})}\\ =&\limsup_{n\to\infty}\frac{2n\log 2n}{9\Big[(1/2)\log(2n\pi)+n\log(n)-n\log e-((4n\log 3)/9)\Big]}\end{aligned}$$

$$=\limsup_{n\to\infty}\frac{2n\log 2n}{9n\log(n)\Big[(1/2\log(n))\log(2n\pi)+1-\log e/\log(n)-((4\log 3)/(9\log(n)))\Big]}=\frac{2}{9}$$

and

$$\begin{aligned}\tau =&\frac{1}{e\rho}\ \limsup_{n\to\infty}\left(2n\|c_n\|^{\rho/(2n)}\right)\\ =&\frac{9}{2e}\ \limsup_{n\to\infty} 2n\left((\frac{1}{n!})^9\ 3^{4n}\right)^{1/(9n)} = 9\sqrt[9]{81}.\end{aligned}$$

Applying Theorem 3.1 and Theorem 3.2 lead immediately to

$$\rho=\frac{\rho_1\rho_2}{\alpha\rho_2+\beta\rho_1}=\frac{2\times 2}{5\times 2+4\times 2}=\frac{2}{9}$$

and

$$\tau=\frac{1}{\rho}(\rho_1\tau_1)^{\alpha\rho/\rho_1}\,(\rho_2\tau_2)^{\beta\rho/\rho_2}=\frac{9}{2}\times 2^{5\times(2/(18))}\times 6^{4\times(2/18)}=9\sqrt[9]{81}.$$

Setting $\alpha=\beta=1$ in (3.4) and (3.5) we get the following corollary concerning the Hadamard product of entire axially monogenic function.

3.4. Corollary. *The Hadamard product of entire axially monogenic function* $\mathcal{H}(x)=(\mathcal{G}_1\triangle\mathcal{G}_2)(x;1,1)$ *is of growth order* ρ *does not exceed* $\frac{\rho_1\rho_2}{\rho_2+\rho_1}$ *and in the case of equality it is of growth type* τ *does not exceed* $\frac{1}{\rho}(\rho_1\tau_1)^{\rho/\rho_1}\,(\rho_2\tau_2)^{\rho/\rho_2}$.

Moreover, if $\alpha+\beta=1$, we obtain the following result

3.5. Corollary. *The generalized Hadamard product of entire axially monogenic function* $\mathcal{H}(x)=(\mathcal{G}_1\triangle\mathcal{G}_2)(x;\alpha,1-\alpha)$ *is of growth order*

$$\rho\le\frac{\rho_1\rho_2}{\alpha(\rho_2-\rho_1)+\rho_1};\quad 0\le\alpha\le 1,$$

and of growth type

$$\tau\le\frac{1}{\rho}(\rho_1\tau_1)^{\alpha\rho/\rho_1}\,(\rho_2\tau_2)^{(1-\alpha)\rho/\rho_2}.$$

Under condition that

$$\rho=\frac{\rho_1\rho_2}{\alpha(\rho_2-\rho_1)+\rho_1};\quad 0\le\alpha\le 1.$$

4. Linear substitution

Let $\mathcal{G}_s(x)=\sum_{n=0}^{\infty}z_n(x)a_{n,s},\quad s=1,2$ are two entire axially monogenic functions of finite positive growth orders ρ_s and growth types τ_s, then the axially monogenic functions $\mathcal{G}_s^*(x)=\mathcal{G}_s(x+b)$, where b is any constant, of the same orders $\rho_s^*=\rho_s$ and $\tau_s^*=\tau_s$ (see [3]). So that

$$\rho_s^*=\limsup_{n\to\infty}\frac{(n\log n)}{\log(\|a_{n,s}\|^{-1})} \tag{4.1}$$

and

$$\tau_s^*=\frac{1}{e\rho_s}\limsup_{n\to\infty}n\,\|a_{n,s}\|^{\rho_s/n}. \tag{4.2}$$

For any positive arbitrary small number $\epsilon_s>\rho_s,\ s=1,2$ there exist positive integers $N_s,\ s=1,2$ such that

$$\|a_{n,1}\|^{\alpha}<n^{-\alpha n/\epsilon_1}\ \text{ for } n>N_1\ \text{ and }\ \|a_{n,2}\|^{\beta}<n^{-\beta n/\epsilon_2}\ \text{ for } n>N_2.$$

If the generalized Hadamard product of entire axially monogenic function $\mathcal{H}(x)=(\mathcal{G}_1\triangle\mathcal{G}_2)(x;\alpha,\beta)=\sum_{n=0}^{\infty}\ z_n(x)\ c_n$ is of growth order ρ and growth type τ and the generalized Hadamard product of entire axially monogenic function $\mathcal{H}^*(x+b)=(\mathcal{G}_1\triangle\mathcal{G}_2)(x+b;\alpha,\beta)=\sum_{n=0}^{\infty}\ z_n(x+b)\ c_n$ is of growth order ρ^* and growth type τ^*, then for $N>\max(N_1,N_2)$ we have

$$\|c_n\|<n^{-n(\alpha/\epsilon_1+\beta/\epsilon_2)}\ \text{ for }\ n>N,$$

that is

$$\log(\|c_n\|)^{-1} > n\Big(\alpha/\epsilon_1 + \beta/\epsilon_2\Big)\ \log(n)$$

and

$$\rho^* = \limsup_{n\to\infty} \frac{n\log(n)}{\log(\|c_n\|)^{-1}} \leq \left(\alpha/\epsilon_1 + \beta/\epsilon_2\right)^{-1}.$$

Since ϵ_s can be shown vert near to ρ_s then

$$\rho^* \leq \left(\alpha/\rho_1 + \beta/\rho_2\right)^{-1},$$

it shows that growth orders of the two generalized Hadamard product of entire axially monogenic functions $\mathcal{H}(x)$ and $\mathcal{H}^*(x+b)$ have the same upper bound, in the case that $\rho = \frac{\rho_1\rho_2}{\alpha\rho_2+\beta\rho_1}$, gives that $\rho^* \leq \rho$. On the other hand since $\mathcal{H}(x) = \mathcal{H}^*(x-b)$, then $\rho \leq \rho^*$ therefore $\rho = \rho^*$.

Similarly for the growth type τ^* of the generalized Hadamard product of entire axially monogenic function $\mathcal{H}^*(x+b)$, it follows that

$$\|a_{n,1}\|^\alpha < \Big(\frac{e\rho_1\epsilon_1}{n}\Big)^{n\alpha/\rho_1}$$

and

$$\|a_{n,2}\|^\beta < \Big(\frac{e\rho_2\epsilon_2}{n}\Big)^{n\beta/\rho_2}.$$

Then

$$\|c_n\|^{1/n} < \Big(\frac{e}{n}\Big)^{\beta/\rho_1+\beta/\rho_2}\ (\rho_1\epsilon_1)^{\alpha/\rho_1}\ (\rho_2\epsilon_2)^{\beta/\rho_2}$$

and

$$\begin{aligned}\tau^* =& \frac{1}{e\rho^*}\ \limsup_{n\to\infty} \Big(n\|c_n\|^{\rho^*/n}\Big)\\ \leq& \frac{1}{e\rho}\ \limsup_{n\to\infty} n\Big(\frac{e}{n}\Big)\ (\rho_1\epsilon_1)^{\alpha\rho/\rho_1}\ (\rho_2\epsilon_2)^{\beta\rho/\rho_2},\end{aligned}$$

since ϵ_s can be chosen very close to τ_s, then

$$\tau^* \leq \frac{1}{\rho}\ (\rho_1\tau_1)^{\alpha\rho/\rho_1}\ (\rho_2\tau_2)^{\beta\rho/\rho_2}.$$

In the case that

$$\tau = \frac{1}{\rho}\ (\rho_1\tau_1)^{\alpha\rho/\rho_1}\ (\rho_2\tau_2)^{\beta\rho/\rho_2},$$

we find that $\tau^* \leq \tau$. Again since $\mathcal{H}(x) = \mathcal{H}^*(x-b)$, then $\tau \leq \tau^*$ and thus $\tau^* = \tau$. Therefore the following result follows:

4.1. Theorem. *The two generalized Hadamard product of entire axially monogenic functions $\mathcal{H}(x)$ and $\mathcal{H}^*(x+b)$ have the same upper bound for its growth orders and its growth types and satisfies the same inequalities.*

References

[1] Abul-Ez. M, Hadamard product of bases of polynomials in Clifford analysis, *Complex Variables.*, **43**, 109-128, 2000.

[2] Abul-Ez. M and Constales. D, Basic sets of polynomials in Clifford analysis, *Complex Variables.*, **14**, 177-185, 1990.

[3] Abul-Ez. M and Constales. D, Linear substitution for basic sets of polynomials in Clifford analysis, *Portugaliae Mathematica.*, **48**, 143-154, 1991.

[4] Abul-Ez. M and Constales. D, On convergence properties of basic series representing special monogenic functions, *Arch. Math.*, **81**, 62-71, 2002.

[5] Abul-Ez. M and De Almeida. R, On the lower order and type of entire axially monogenic functions, *Results. Math.,* **63**, 1257-1275, 2013.

[6] Brackx. F, Delanghe. R and Sommen. F, Clifford analysis. Research Notes in Mathematics 76. London: London Pitman Books Ltd, 1982.

[7] Constales. D, De Almeida. R and Krausshar. R, On the relation between the growth and the Taylor coefficients of entire solutions to the higher dimensional Cauchy-Riemann system in R^{n+1}, *J. Math. Anal. Appl.,* **327**, 763-775, 2007.

[8] Constales. D, De Almeida. R and Krausshar. R, On the growth type of entire monogenic functions, *Arch. Math.,* **88**, 153-163, 2007.

[9] Constales. D, De Almeida. R and Krausshar. R, Applications of the maximum term and the central index in the asymptotic growth analysis of entire solutions to higher dimensional polynomial Cauchy-Riemann equations, *Complex Var. Elliptic Equ.,* **53**, 195-213, 2008.

[10] De Almeida. R and Krausshar. R, Basics on growth orders of polymonogenic functions, *Complex Var. Elliptic Equ.,* **60**, 1-25, 2015.

[11] Delanghe. R, Sommen. F and Souccěk. V, Clifford algebra and spinor-valued function. Dordrecht: Kluwer Academic Publishers, 1992.

[12] Dutta. R. K, On order of a function of several complex variables analytic in the unit polydisc, *Krag. J. Math.,* **36**, 163-174, 2012.

[13] Gol'dberg. A. A, Elementary remarks on the formulas defining the order and type entire functions in several variables, *Dokl. Akad. Nauk Arm. SSR.*, **29**, 145-151, 1959.

[14] Jae Hochoi. J and Kim. Y. C, Generalized Hadamard product functions with negative coefficients, *J. Math. Anal. Appl.,* **199**, 459-501, 1996.

[15] Kishka. Z, Abul-Ez. M, Saleem. M and Abd-Elmaged. H, On the order and type of entire matrix functions in complete Reinhardt domain, *J. Mod. Meth. Numer. Math.,* **3**, 31-40, 2012.

[16] Lounesto. P and Bergh. P, Axially symmetric vector fields and their complex potentials, *Complex Variables.,* **2**, 139- 150, 1983.

[17] Ronkin. L. I, Introduction to the theory of entire functions of several variables, *Trans. Math. Monog.,* **44**. Providence R.I., American Mathematical Society, VI, 1974.

[18] Sayyed. K, Metwally. M and Mohamed. M, Some order and type of generalized Hadamard product of entire functions, *South. Asi. Bull. Math.,* **26**, 121-132, 2002.

[19] Sommen. F, Plane elliptic systems and monogenic functions in symmetric domains, *Suppl. Rend. Circ. Mat. Palermo.,* **6**, 259-269, 1984.

[20] Srivastava. R. K and Kumar. V, On the order and type of integral functions of several complex variables, *Compo. Math.*, **17**, 161-166, 1966.

[21] Srivastava. G. S and Kumar. S, On the generalized order and generalized type of entire monogenic functions, *Demon. Math.,* **46**, 663-677, 2013.