Hacettepe Journal of Mathematics and Statistics
Volume 47(5) (2018), 1375 – 1393

Improved ratio estimators using some robust measures

Muhammad Abid*[†], Hafiz Zafar Nazir[‡], Muhammad Riaz[§], Zhengyan Lin[¶] and Hafiz Muhammad Tahir[‖]

Abstract

Estimation of population mean is of prime concern in many studies and ratio estimators are popular choices for it. It is a common practice to use conventional measures of location to develop ratio estimators using information on auxiliary variables. In this article, we propose a class of ratio estimators for a finite population mean using information on two auxiliary variables with the help of some non-conventional location measures. We have incorporated tri-mean, Hodges-Lehmann, mid-range and decile mean of the two auxiliary variables to serve the purpose. The properties associated with the proposed class of ratio estimators are evaluated using mean square error. We have presented efficiency comparisons of the proposed class of ratio estimators with other existing estimators under the optimal conditions. An empirical study is also included for illustration and verification purposes. From theoretical and empirical study, we observed that the proposed estimators perform better as compared to the usual ratio and the existing estimators consider in this study.

Keywords: Auxiliary variables, Hodges-Lehmann estimator, Mean square error, Mid-range, Ratio estimators, Tri-mean.

Received : 22/07/2016 *Accepted* : 31/01/2017 *Doi* : 10.15672/HJMS.2017.442

---

*Department of Mathematics, Institute of Statistics, Zhejiang University, Hangzhou, 310027, China. Department of Statistics, Faculty of Science And Technology, Government College University, Faisalabad, Pakistan, Email: `mabid@zju.edu.cn`

[†] Corresponding Author.

[‡] Department of Statistics, Faculty of Science, University of Sargodha, Pakistan, Email: `hafizzafarnazir@yahoo.com`

[§] Department of Mathematics and Statistics, King Fahad University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia, Email: `riaz76qau@yahoo.com`

[¶] Department of Mathematics, Institute of Statistics, Zhejiang University, Hangzhou, 310027, China, Email: `zlin@zju.edu.cn`

[‖] Department of Statistics, Faculty of Science And Technology, Government College University, Faisalabad, Pakistan, Email: `tahirqaustat@yahoo.com`

1. Introduction

In practice, we may come across different variables that offer information on every unit in the population. These variables are classified in two types namely variables of interest and auxiliary variables. The former are of direct interest in a study and are named study variables, while the later are instead employed to improve the sampling plan or to enhance estimation of the study variables. The auxiliary variables are generally associated with the study variables and we may use this information in different forms such as ratio, product and regression to mention a few etc. The auxiliary information may be available from different sources such as similar studies in past, economic reports, national census etc.

The ratio and regression estimators are used to improve the efficiency of the simple random sampling without replacement (SRSWOR) sample mean when there is a positive correlation exist between study variable (the variable of direct interst) and an auxiliary variable under certain conditions (see for example Cochran [7] and Murthy [18]). When the population parameters of the auxiliary variable, such as population mean, kurtosis, skewness, coefficient of variation, median, quartiles, correlation coefficient, deciles etc., are known, ratio estimators and their modifications are available in the literature which perform better than the usual sample mean under the SRSWOR. For further details on the modified ratio estimators, readers are referred to Abid et al. [1], [2], [3] and [4], Subramani and Kumarapandiyan [29], [30], [31], [32] and [33], Yan and Tian [37], Kadilar and Cingi [10] and [13], Upadhyaya and Singh [35], and Sisodia and Dwivedi [26] and reference therein.

Consider a finite population $Z = \{Z_1, Z_2, Z_3, \dots, Z_N\}$ of N distinct and identifiable units. Let Y, X_1 and X_2 be the study variable and the auxiliary variables with corresponding values Y_i, X_{1i} and X_{2i}, respectively, for the ith unit $i = \{1, 2, \dots, N\}$ defined on a finite population Z. Let $X_{1(1)}, X_{1(2)}, \dots, X_{1(N)}$ and $X_{2(1)}, X_{2(2)}, \dots, X_{2(N)}$ be the order statistics of two auxiliary variables, X_1 and X_2. The objective is to estimate population mean $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ on the basis of a random sample by using two auxiliary variables.

The notations used in this paper can be described as follows:

NOMENCLATURE

Romen

N, Population size

n, Sample size

$f = n/N$, Sampling fraction

Y, Study variable

X_1, X_2 , Auxiliary variables

$\bar{X}_1, \bar{X}_2, \bar{Y}$, Population means of auxiliary variables and study variable

$\bar{x}_1, \bar{x}_2, \bar{y}$, Sample means of auxiliary variables and study variable

$C_{x1}, C_{x2},\ C_y$, Coefficient of variation of auxiliary variables and study variable

$MSE(.)$, Mean square error of the estimator

$\hat{\bar{Y}}_i$, Existing ratio estimators based on two auxiliary variables of $\bar{Y}$,

$\hat{\bar{Y}}_{pj}$, Proposed ratio estimators based on two auxiliary variables of $\bar{Y}$

Subscript

i, For existing estimators

j, For proposed estimators

Greek

ρ_{yx_1},ρ_{yx_2},$\rho_{x_1x_2}$, Coefficient of correlation between study and auxiliary variables $\beta_{2(x_1)} = \frac{N(N+1)\sum_{i=1}^{N}(X_i-\bar{X}_1)^4}{(N-1)(N-2)(N-3)S^4} - \frac{3(N-1)^2}{(N-2)(N-3)}$, $\beta_{2(x_2)} = \frac{N(N+1)\sum_{i=1}^{N}(X_i-\bar{X}_2)^4}{(N-1)(N-2)(N-3)S^4} - \frac{3(N-1)^2}{(N-2)(N-3)}$, Coefficient of kurtosis of two auxiliary variables.

Based on mentioned notations, the usual ratio multivariate estimator using information on two auxiliary variables X_1 and X_2 to estimate the population mean, $\bar{Y}$, as follows:

$$\hat{\bar{Y}}_{MR} = \gamma_1 \bar{y}\,\frac{\bar{X}_1}{\bar{x}_1} + \gamma_2 \bar{y}\frac{\bar{X}_2}{\bar{x}_2}$$

where γ_1 and γ_2 are the weights which satisfy the condition $\gamma_1 + \gamma_2 = 1$ and $\bar{x}_1, \bar{x}_2$ and $\bar{X}_1, \bar{X}_2$ are, respectively, the sample and populations means of auxiliary variables. The mean squared error (MSE) of the usual ratio estimator based on two auxiliary variables is given by The mean squared error (MSE) of the usual ratio estimator based on two auxiliary variables is given by

$$MSE\left(\hat{\bar{Y}}_{MR}\right) \cong \frac{1-f}{n}\bar{Y}^2(C_y^2 + \gamma_1^2 C_{x_1}^2 + \gamma_2^2 C_{x_2}^2 - 2\gamma_1\rho_{yx_1}C_yC_{x1} \\ -2\gamma_2\rho_{yx_2}C_yC_{x2} + 2\gamma_1\gamma_2\rho_{x_1x_2}C_{x1}C_{x2}) \tag{1.1}$$

The optimum values of γ_1 and γ_2 are given by

$$\gamma_1^* = \frac{C_{x_2}^2 + \rho_{yx_1}C_yC_{x1} - \rho_{yx_2}C_yC_{x2} - \rho_{x_1x_2}C_{x1}C_{x2}}{C_{x_1}^2 + C_{x_2}^2 - 2\rho_{x_1x_2}C_{x1}C_{x2}}, \gamma_2^* = 1 - \gamma_1^*$$

So, the minimum MSE is

$$MSE_{min}\left(\hat{\bar{Y}}_{MR}\right) \cong \frac{1-f}{n}\bar{Y}^2(C_y^2 + \gamma_1^{*2} C_{x_1}^2 + \gamma_2^{*2} C_{x_2}^2 - 2\gamma_1^*\rho_{yx_1}C_yC_{x1} \\ -2\gamma_2^*\rho_{yx_2}C_yC_{x2} + 2\gamma_1^*\gamma_2^*\rho_{x_1x_2}C_{x1}C_{x2}) \tag{1.2}$$

Knowledge of two auxiliary variables in the framework of ratio estimators is used in this paper. Using information on two auxiliary variables several modified ratio estimators have been proposed by linking together ratio estimators, product estimators and regression estimators in order to find better results. For a more detailed discussion on the ratio estimator and its modifications using two auxiliary variables , one may refer to Lu and Yan [16], Lu et al. [17], Al-Hossain and Khan [6], Subramani and Prabavathy [34], Lu [15], Khare et al. [14], Perri [20], Kadilar and Cingi [11] and [12], Singh and Tailor [24] and [25], Abu-Dayyeh et al. [5], Srivastava and Jhajj [28], Srivastava [27], Raj [21] and Olkin [19].

The organization of the rest of the article is as follows: Section 2 provides a description of the existing estimators. The structure of proposed ratio estimator based on two auxiliary variables is given in Section 3. The efficiency comparisons of the proposed estimator with the existing estimator are presented in Section 4. Section 5, consists of an empirical study of proposed estimators. Finally, Section 6 summarizes the findings of this study.

2. Existing ratio estimators

Singh [23] proposed a ratio estimator using information on two auxiliary variables for estimating the population mean $\bar{Y}$ as follows:

$$\hat{\bar{Y}}_S = \bar{y}\left(\frac{\bar{X}_1}{\bar{x}_1}\right)\left(\frac{\bar{X}_2}{\bar{x}_2}\right)$$

The MSE of the estimator suggested by Singh [23] is given below:

$$MSE\left(\hat{\bar{Y}}_S\right) \cong \frac{1-f}{n}\bar{Y}^2\left(C_y^2 + C_{x_1}^2 + C_{x_2}^2 - 2\rho_{yx_1}C_yC_{x1} + 2\rho_{x_1x_2}C_{x1}C_{x2}\right) \tag{2.1}$$

Using the known value of correlation coefficient of two auxiliary variables Singh and Tailor [25] suggested the following modified ratio cum product estimator

$$\hat{\bar{Y}}_{ST} = \bar{y}\left(\frac{\bar{X}_1 + \rho_{x_1x_2}}{\bar{x}_1 + \rho_{x_1x_2}}\right)\left(\frac{\bar{x}_2 + \rho_{x_1x_2}}{\bar{X}_2 + \rho_{x_1x_2}}\right)$$

The MSE of Singh and Tailor [25] proposed estimator is given as

$$\begin{aligned} MSE\left(\hat{\bar{Y}}_{ST}\right) \cong & \frac{1-f}{n}\bar{Y}^2(C_y^2 + \delta_1^* C_{x_1}^2(\delta_1^* - 2k_{yx1}) \\ & + \delta_2^* C_{x_2}^2(\delta_2^* + 2(k_{yx2} - \delta_1^* k_{x1x2}) \end{aligned} \tag{2.2}$$

where, $k_{yx1} = \rho_{yx_1}\frac{C_y}{C_{x1}}$, $k_{yx2} = \rho_{yx_2}\frac{C_y}{C_{x2}}$, $k_{x1x2} = \rho_{x_1x_2}\frac{C_{x1}}{C_{x2}}$, $\delta_1^* = \frac{\bar{X}_1}{\bar{X}_1 + \rho_{x_1x_2}}$ and $\delta_2^* = \frac{\bar{X}_2}{\bar{X}_2 + \rho_{x_1x_2}}$.

Lu and Yan [16] proposed the ratio estimators by using the known values of correlation coefficient, coefficient of variation and coefficient of kurtosis of two auxiliary variables. They showed that their proposed estimator performs efficiently as compared to the estimators suggested by Abu-Dayyeh et al. [5] and usual ratio estimator based on two auxiliary variables. The Lu and Yan [16] proposed the following estimators

$$\begin{aligned}
\hat{\bar{Y}}_1 &= a_1\bar{y}\left(\frac{\bar{X}_1 + C_{x1}}{\bar{x}_1 + C_{x1}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2 + C_{x2}}{\bar{x}_2 + C_{x2}}\right) \\
\hat{\bar{Y}}_2 &= a_1\bar{y}\left(\frac{\bar{X}_1 + \beta_{2(x_1)}}{\bar{x}_1 + \beta_{2(x_1)}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2 + \beta_{2(x_2)}}{\bar{x}_2 + \beta_{2(x_2)}}\right) \\
\hat{\bar{Y}}_3 &= a_1\bar{y}\left(\frac{\bar{X}_1\beta_{2(x_1)} + C_{x1}}{\bar{x}_1\beta_{2(x_1)} + C_{x1}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2\beta_{2(x_2)} + C_{x2}}{\bar{x}_2\beta_{2(x_2)} + C_{x2}}\right) \\
\hat{\bar{Y}}_4 &= a_1\bar{y}\left(\frac{\bar{X}_1C_{x1} + \beta_{2(x_1)}}{\bar{x}_1C_{x1} + \beta_{2(x_1)}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2C_{x2} + \beta_{2(x_2)}}{\bar{x}_2C_{x2} + \beta_{2(x_2)}}\right) \\
\hat{\bar{Y}}_5 &= a_1\bar{y}\left(\frac{\bar{X}_1 + \rho_{yx_1}}{\bar{x}_1 + \rho_{yx_1}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2 + \rho_{yx_2}}{\bar{x}_2 + \rho_{yx_2}}\right) \\
\hat{\bar{Y}}_6 &= a_1\bar{y}\left(\frac{\bar{X}_1C_{x1} + \rho_{yx_1}}{\bar{x}_1C_{x1} + \rho_{yx_1}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2C_{x2} + \rho_{yx_2}}{\bar{x}_2C_{x2} + \rho_{yx_2}}\right) \\
\hat{\bar{Y}}_7 &= a_1\bar{y}\left(\frac{\bar{X}_1\rho_{yx_1} + C_{x1}}{\bar{x}_1\rho_{yx_1} + C_{x1}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2\rho_{yx_2} + C_{x2}}{\bar{x}_2\rho_{yx_2} + C_{x2}}\right) \\
\hat{\bar{Y}}_8 &= a_1\bar{y}\left(\frac{\bar{X}_1\beta_{2(x_1)} + \rho_{yx_1}}{\bar{x}_1\beta_{2(x_1)} + \rho_{yx_1}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2\beta_{2(x_2)} + \rho_{yx_2}}{\bar{x}_2\beta_{2(x_2)} + \rho_{yx_2}}\right) \\
\hat{\bar{Y}}_9 &= a_1\bar{y}\left(\frac{\bar{X}_1\rho_{yx_1} + \beta_{2(x_1)}}{\bar{x}_1\rho_{yx_1} + \hat{a}_{2(x_1)}}\right) + a_2\bar{y}\left(\frac{\bar{X}_2\rho_{yx_2} + \beta_{2(x_2)}}{\bar{x}_2\rho_{yx_2} + \beta_{2(x_2)}}\right)
\end{aligned}$$

The MSE of Lu and Yan [16] proposed estimators are given as

$$\begin{aligned} MSE\left(\hat{\bar{Y}}_i\right) \cong & \frac{1-f}{n}\bar{Y}^2(C_y^2 + a_1^2R_{1i}^2C_{x_1}^2 + a_2^2R_{2i}^2C_{x_2}^2 \\ & -2a_1R_{1i}\rho_{yx_1}C_yC_{x1} - 2a_2R_{2i}\rho_{yx_2}C_yC_{x2} + 2a_1a_2R_{1i}R_{2i}\rho_{x_1x_2}C_{x1}C_{x2}) \end{aligned} \tag{2.3}$$

where, $i = 1, 2, \ldots, 9$.

The optimum values of a_1 and a_2 can easily be found by differentiating equation (2.3) with respect to a_1 and a_2 and equating it equal to zero. The optimum values of a_1 and a_2 are

$$a_1^* = \frac{R_{2i}^2C_{x_2}^2 + R_{1i}\rho_{yx_1}C_yC_{x1} - R_{1i}R_{2i}\rho_{x_1x_2}C_{x1}C_{x2} - R_{2i}\rho_{yx_2}C_yC_{x2}}{R_{1i}^2C_{x_1}^2 - 2R_{1i}R_{2i}\rho_{x_1x_2}C_{x1}C_{x2} + R_{2i}^2C_{x_2}^2},$$

$a_2^* = 1 - k_1^*$.

Hence, the minimum MSE of Lu and Yan [16] estimators are given by

$$MSE_{min}\left(\hat{\bar{Y}}_i\right) \cong \frac{1-f}{n}\bar{Y}^2 (C_y^2 + a_1^{*2}R_{1i}^2C_{x_1}^2 + a_2^{*2}R_{2i}^2C_{x_2}^2 \tag{2.4}$$

$$-2a_1^*R_{1i}\rho_{yx_1}C_yC_{x1} - 2a_2^*R_{2i}\rho_{yx_2}C_yC_{x2} + 2a_1^*a_2^*R_{1i}R_{2i}\rho_{x_1x_2}C_{x1}C_{x2})$$

where $i = 1, 2, \ldots, 9$ and the values of constants R_{1i} and R_{2i} are,

$$R_{11} = \left(\frac{\bar{X}_1}{\bar{X}_1 + C_{x1}}\right), R_{12} = \left(\frac{\bar{X}_1}{\bar{X}_1 + \beta_{2(x_1)}}\right),$$

$$R_{13} = \left(\frac{\bar{X}_1\beta_{2(x_1)}}{\bar{X}_1\beta_{2(x_1)} + C_{x1}}\right), R_{14} = \left(\frac{\bar{X}_1C_{x1}}{\bar{X}_1C_{x1} + \beta_{2(x_1)}}\right),$$

$$R_{15} = \left(\frac{\bar{X}_1}{\bar{X}_1 + \rho_{yx_1}}\right), R_{16} = \left(\frac{\bar{X}_1C_{x1}}{\bar{X}_1C_{x1} + \rho_{yx_1}}\right),$$

$$R_{17} = \left(\frac{\bar{X}_1\rho_{yx_1}}{\bar{X}_1\rho_{yx_1} + C_{x1}}\right), R_{18} = \left(\frac{\bar{X}_1\beta_{2(x_1)}}{\bar{X}_1\beta_{2(x_1)} + \rho_{yx_1}}\right),$$

$$R_{19} = \left(\frac{\bar{X}_1\rho_{yx_1}}{\bar{X}_1\rho_{yx_1} + \beta_{2(x_1)}}\right), R_{21} = \left(\frac{\bar{X}_2}{\bar{X}_2 + C_{x2}}\right),$$

$$R_{22} = \left(\frac{\bar{X}_2}{\bar{X}_2 + \beta_{2(x_2)}}\right), R_{23} = \left(\frac{\bar{X}_2\beta_{2(x_2)}}{\bar{X}_2\beta_{2(x_2)} + C_{x2}}\right),$$

$$R_{24} = \left(\frac{\bar{X}_2C_{x2}}{\bar{X}_2C_{x2} + \beta_{2(x_2)}}\right), R_{25} = \left(\frac{\bar{X}_2}{\bar{X}_2 + \rho_{yx_2}}\right),$$

$$R_{26} = \left(\frac{\bar{X}_2C_{x2}}{\bar{X}_2C_{x2} + \rho_{yx_2}}\right), R_{27} = \left(\frac{\bar{X}_2\rho_{yx_2}}{\bar{X}_2\rho_{yx_2} + C_{x2}}\right),$$

$$R_{28} = \left(\frac{\bar{X}_2\beta_{2(x_2)}}{\bar{X}_2\beta_{2(x_2)} + \rho_{yx_2}}\right), R_{29} = \left(\frac{\bar{X}_2\rho_{yx_2}}{\bar{X}_2\rho_{yx_2} + \beta_{2(x_2)}}\right).$$

3. Proposed class of ratio estimators

In this section, we propose different ratio type estimators using the known information on population tri-mean, mid-range, Hodges-Lehmann, decile mean, coefficient of variation, coefficient of kurtosis and correlation coefficient of two auxiliary variables. The mid-range defined as: $MR_1 = \frac{X_{1(1)} + X_{1(N)}}{2}$, and $MR_2 = \frac{X_{2(1)} + X_{2(N)}}{2}$, where $X_{1(1)}$ and $X_{1(N)}$ are the lowest and highest order statistics in a population of size N for X_1 and $X_{2(1)}$ and $X_{2(N)}$ are the lowest and highest order statistics in a population of size N for X_2. It is highly sensitive to outliers as its design structure is based on only extreme values of data (cf. Ferrell [8] for more details). We also include the measure based on the median of the pairwise Walsh averages which is known as Hodges-Lehmann (HL) estimator. The HL estimator is defined as: $HL_1 = \text{median}\left(\left(X_{1(l)} + X_{1(k)}\right)/2, 1 \leq l \leq k \leq N\right)$, and $HL_2 = \text{median}\left(\left(X_{2(l)} + X_{2(k)}\right)/2, 1 \leq l \leq k \leq N\right)$ for two auxiliary variables. The main advantage of the HL is that it is robust against outliers. For more properties of HL (see Hettmansperger and McKean [9]). The next measure included in this study is the tri-mean (TM), which is the weighted average of the population median and two quartiles and is defined as: $TM_1 = \frac{Q_{1(1)} + 2Q_{1(2)} + Q_{1(3)}}{4}$, and $TM_2 = \frac{Q_{2(1)} + 2Q_{2(2)} + Q_{2(3)}}{4}$, where $Q_{1(p)}$ $(p = 1, 2, 3)$ denote one of the three quartiles in a population for X_1 and $Q_{2(p)}$ $(p = 1, 2, 3)$ denote one of the three quartiles in a population for X_2. For detailed properties of TM (see Wang et al. [36] and Abid et al. [1]). The last measure include in this study is the decile mean (DM) which is defined as: $DM_1 = \frac{D_{1(1)} + D_{1(2)} + \cdots + D_{1(9)}}{9}$, and $DM_2 = \frac{D_{2(1)}+D_{2(2)}+\cdots+D_{2(9)}}{9}$,where $D_{1(1)} + D_{1(2)} + \cdots + D_{1(9)}$ and $D_{2(1)} + D_{2(2)} + \cdots + D_{2(9)}$ are the deciles for X_1 and X_2, respectively. The main advantage of the DM is that it is also less sensitive to extreme values than any other existing measures as well as it depends on the eighty percent of a sample, a population, or a probability distribution. So, it is also referred as a robust measure in this regard (see Abid et al. [2] and [4] for more detail).

The proposed ratio estimators based on two auxiliary variables are given below;

$$\hat{\bar{Y}}_{p1} = k_1\bar{y}\left(\frac{\bar{X}_1 + MR_1}{\bar{x}_1 + MR_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2 + MR_2}{\bar{x}_2 + MR_2}\right)$$

$$\hat{\bar{Y}}_{p2} = k_1\bar{y}\left(\frac{\bar{X}_1 + TM_1}{\bar{x}_1 + TM_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2 + TM_2}{\bar{x}_2 + TM_2}\right)$$

$$\hat{\bar{Y}}_{p3} = k_1\bar{y}\left(\frac{\bar{X}_1 + HL_1}{\bar{x}_1 + HL_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2 + HL_2}{\bar{x}_2 + HL_2}\right)$$

$$\hat{\bar{Y}}_{p4} = k_1\bar{y}\left(\frac{\bar{X}_1 + DM_1}{\bar{x}_1 + DM_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2 + DM_2}{\bar{x}_2 + DM_2}\right)$$

$$\hat{\bar{Y}}_{p5} = k_1\bar{y}\left(\frac{\bar{X}_1\beta_{2(x_1)} + MR_1}{\bar{x}_1\beta_{2(x_1)} + MR_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2\beta_{2(x_2)} + MR_2}{\bar{x}_2\beta_{2(x_2)} + MR_2}\right)$$

$$\hat{\bar{Y}}_{p6} = k_1\bar{y}\left(\frac{\bar{X}_1\beta_{2(x_1)} + TM_1}{\bar{x}_1\beta_{2(x_1)} + TM_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2\beta_{2(x_2)} + TM_2}{\bar{x}_2\beta_{2(x_2)} + TM_2}\right)$$

$$\hat{\bar{Y}}_{p7} = k_1\bar{y}\left(\frac{\bar{X}_1\beta_{2(x_1)} + HL_1}{\bar{x}_1\beta_{2(x_1)} + HL_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2\beta_{2(x_2)} + HL_2}{\bar{x}_2\beta_{2(x_2)} + HL_2}\right)$$

$$\hat{\bar{Y}}_{p8} = k_1\bar{y}\left(\frac{\bar{X}_1\beta_{2(x_1)} + DM_1}{\bar{x}_1\beta_{2(x_1)} + DM_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2\beta_{2(x_2)} + DM_2}{\bar{x}_2\beta_{2(x_2)} + DM_2}\right)$$

$$\hat{\bar{Y}}_{p9} = k_1\bar{y}\left(\frac{\bar{X}_1C_{x1} + MR_1}{\bar{x}_1C_{x1} + MR_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2C_{x2} + MR_2}{\bar{x}_2C_{x2} + MR_2}\right)$$

$$\hat{\bar{Y}}_{p10} = k_1\bar{y}\left(\frac{\bar{X}_1C_{x1} + TM_1}{\bar{x}_1C_{x1} + TM_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2C_{x2} + TM_2}{\bar{x}_2C_{x2} + TM_2}\right)$$

$$\hat{\bar{Y}}_{p11} = k_1\bar{y}\left(\frac{\bar{X}_1C_{x1} + HL_1}{\bar{x}_1C_{x1} + HL_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2C_{x2} + HL_2}{\bar{x}_2C_{x2} + HL_2}\right)$$

$$\hat{\bar{Y}}_{p12} = k_1\bar{y}\left(\frac{\bar{X}_1C_{x1} + DM_1}{\bar{x}_1C_{x1} + DM_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2C_{x2} + DM_2}{\bar{x}_2C_{x2} + DM_2}\right)$$

$$\hat{\bar{Y}}_{p13} = k_1\bar{y}\left(\frac{\bar{X}_1\rho_{yx_1} + MR_1}{\bar{x}_1\rho_{yx_1} + MR_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2\rho_{yx_2} + MR_2}{\bar{x}_2\rho_{yx_2} + MR_2}\right)$$

$$\hat{\bar{Y}}_{p14} = k_1\bar{y}\left(\frac{\bar{X}_1\rho_{yx_1} + TM_1}{\bar{x}_1\rho_{yx_1} + TM_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2\rho_{yx_2} + TM_2}{\bar{x}_2\rho_{yx_2} + TM_2}\right)$$

$$\hat{\bar{Y}}_{p15} = k_1\bar{y}\left(\frac{\bar{X}_1\rho_{yx_1} + HL_1}{\bar{x}_1\rho_{yx_1} + HL_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2\rho_{yx_2} + HL_2}{\bar{x}_2\rho_{yx_2} + HL_2}\right)$$

$$\hat{\bar{Y}}_{p16} = k_1\bar{y}\left(\frac{\bar{X}_1\rho_{yx_1} + DM_1}{\bar{x}_1\rho_{yx_1} + DM_1}\right) + k_2\bar{y}\left(\frac{\bar{X}_2\rho_{yx_2} + DM_2}{\bar{x}_2\rho_{yx_2} + DM_2}\right)$$

The MSE of the proposed estimators are given as

$$MSE\left(\hat{\bar{Y}}_{p}j\right) \cong \frac{1-f}{n}\bar{Y}^2(C_y^2 + k_1^2R_{p1j}^2C_{x_1}^2 + k_2^2R_{p2j}^2C_{x2}^2 \tag{3.1}$$
$$-2k_1R_{p1j}\rho_{yx_1}C_yC_{x1} - 2k_2R_{p2j}\rho_{yx_2}C_yC_{x2} + 2k_1k_2R_{p1j}R_{p2j}\rho_{x_1x_2}C_{x1}C_{x2})$$

where, $j = 1, 2, \ldots, 16$. The optimum values of k_1 and k_2 for the proposed ratio estimators can easily be found by differentiating equation (3.1) with respect to k_1 and k_2 and equating it equal to zero. The optimum values of k_1 and k_2 are

$$k_1^* = \frac{R_{p2j}^2 C_{x_2}^2 + R_{p1j}\rho_{yx_1} C_y C_{x1} - R_{p1j} R_{p2j} \rho_{x_1 x_2} C_{x1} C_{x2} - R_{p2j} \rho_{yx_2} C_y C_{x2}}{R_{p1j}^2 C_{x_1}^2 - 2 R_{p1j} R_{p2j} \rho_{x_1 x_2} C_{x1} C_{x2} + R_{p2j}^2 C_{x_2}^2},$$

$$k_2^* = 1 - k_1^*.$$

Hence, the minimum MSE of the proposed ratio estimators are given by

$$MSE_{min}\left(\hat{\bar{Y}}_{pj}\right) \cong \frac{1-f}{n} \bar{Y}^2 (C_y^2 + k_1^{*2} R_{p1j}^2 C_{x_1}^2 + k_2^{*2} R_{p2j}^2 C_{x_2}^2 \tag{3.2}$$
$$-2k_1^* R_{p1j} \rho_{yx_1} C_y C_{x1} - 2k_2^* R_{p2j} \rho_{yx_2} C_y C_{x2} + 2k_1^* k_2^* R_{p1j} R_{p2j} \rho_{x_1 x_2} C_{x1} C_{x2}$$

where, $j = 1, 2, \ldots, 16$ and the values of constant R_{p1j} and R_{p2j} are,

$$R_{p11} = \left(\frac{\bar{X}_1}{\bar{X}_1 + MR_1}\right), R_{p12} = \left(\frac{\bar{X}_1}{\bar{X}_1 + TM_1}\right),$$

$$R_{p13} = \left(\frac{\bar{X}_1}{\bar{X}_1 + HL_1}\right), R_{p14} = \left(\frac{\bar{X}_1}{\bar{X}_1 + DM_1}\right),$$

$$R_{p15} = \left(\frac{\bar{X}_1 \beta_{2(x_1)}}{\bar{X}_1 \beta_{2(x_1)} + MR_1}\right), R_{p16} = \left(\frac{\bar{X}_1 \beta_{2(x_1)}}{\bar{X}_1 \beta_{2(x_1)} + TM_1}\right),$$

$$R_{p17} = \left(\frac{\bar{X}_1 \beta_{2(x_1)}}{\bar{X}_1 \beta_{2(x_1)} + HL_1}\right), R_{p18} = \left(\frac{\bar{X}_1 \beta_{2(x_1)}}{\bar{X}_1 \beta_{2(x_1)} + DM_1}\right),$$

$$R_{p19} = \left(\frac{\bar{X}_1 C_{x1}}{\bar{X}_1 C_{x1} + MR_1}\right), R_{p110} = \left(\frac{\bar{X}_1 C_{x1}}{\bar{X}_1 C_{x1} + TM_1}\right),$$

$$R_{p111} = \left(\frac{\bar{X}_1 C_{x1}}{\bar{X}_1 C_{x1} + HL_1}\right), R_{p112} = \left(\frac{\bar{X}_1 C_{x1}}{\bar{X}_1 C_{x1} + DM_1}\right),$$

$$R_{p113} = \left(\frac{\bar{X}_1 \rho_{yx_1}}{\bar{X}_1 \rho_{yx_1} + MR_1}\right), R_{p114} = \left(\frac{\bar{X}_1 \rho_{yx_1}}{\bar{X}_1 \rho_{yx_1} + TM_1}\right),$$

$$R_{p115} = \left(\frac{\bar{X}_1 \rho_{yx_1}}{\bar{X}_1 \rho_{yx_1} + HL_1}\right), R_{p116} = \left(\frac{\bar{X}_1 \rho_{yx_1}}{\bar{X}_1 \rho_{yx_1} + DM_1}\right).$$

$$R_{p21} = \left(\frac{\bar{X}_2}{\bar{X}_2 + MR_2}\right), R_{p22} = \left(\frac{\bar{X}_2}{\bar{X}_2 + TM_2}\right),$$

$$R_{p23} = \left(\frac{\bar{X}_2}{\bar{X}_2 + HL_2}\right), R_{p24} = \left(\frac{\bar{X}_2}{\bar{X}_2 + DM_2}\right),$$

$$R_{p25} = \left(\frac{\bar{X}_2 \beta_{2(x_2)}}{\bar{X}_2 \beta_{2(x_2)} + MR_2}\right), R_{p26} = \left(\frac{\bar{X}_2 \beta_{2(x_2)}}{\bar{X}_2 \beta_{2(x_2)} + TM_2}\right),$$

$$R_{p27} = \left(\frac{\bar{X}_2 \beta_{2(x_2)}}{\bar{X}_2 \beta_{2(x_2)} + HL_2}\right), R_{p28} = \left(\frac{\bar{X}_2 \beta_{2(x_2)}}{\bar{X}_2 \beta_{2(x_2)} + DM_2}\right),$$

$$R_{p29} = \left(\frac{\bar{X}_2 C_{x2}}{\bar{X}_2 C_{x2} + MR_2}\right), R_{p210} = \left(\frac{\bar{X}_2 C_{x2}}{\bar{X}_2 C_{x2} + TM_2}\right),$$

$$R_{p211} = \left(\frac{\bar{X}_2 C_{x2}}{\bar{X}_2 C_{x2} + HL_2}\right), R_{p212} = \left(\frac{\bar{X}_2 C_{x2}}{\bar{X}_2 C_{x2} + DM_2}\right),$$

$$R_{p213} = \left(\frac{\bar{X}_2 \rho_{yx_2}}{\bar{X}_2 \rho_{yx_2} + MR_2}\right), R_{p214} = \left(\frac{\bar{X}_2 \rho_{yx_2}}{\bar{X}_2 \rho_{yx_2} + TM_2}\right),$$

$$R_{p215}=\left(\frac{\bar{X}_2\rho_{yx_2}}{\bar{X}_2\rho_{yx_2}+HL_2}\right),R_{p216}=\left(\frac{\bar{X}_2\rho_{yx_2}}{\bar{X}_2\rho_{yx_2}+DM_2}\right).$$

It is to be noted that Lu and Yan [16] and the proposed estimators using information of two auxiliary variables are belongs to the following general class of ratio estimators for $\bar{Y}$ defined as (cf. Lu and Yan [16])

$$\hat{\bar{Y}}_{gc}=K_1\bar{y}\left(\frac{T_1\bar{X}_1+P_1}{T_1\bar{x}_1+P_1}\right)+K_2\bar{y}\left(\frac{T_2\bar{X}_2+P_2}{T_2\bar{x}_2+P_2}\right)$$

where $(K_1,\ K_2)$ are weights that satisfy the condition, $K_1+K_2=1$, $T_1\,(\neq 0)$, $T_2\,(\neq 0)$, $P_1,\ P_2$ are either constant or function of known parameters of the population.

To the first degree of approximation the MSE of general class of ratio estimators for $\bar{Y}$ can be obtained as follows:

Let us define, $e_0=\frac{\bar{y}-\bar{Y}}{\bar{Y}}$, $e_1=\frac{\bar{x}_1-\bar{X}_1}{\bar{X}_1}$, $e_2=\frac{\bar{x}_2-\bar{X}_2}{\bar{X}_2}$, then $\bar{y}=\bar{Y}(1+e_0)$, $\bar{x}_1=\bar{X}_1\left(1+e_1\right)$, and $\bar{x}_2=\bar{X}_2\left(1+e_2\right)$. From the definition of $e_0,\ e_1$ and e_2, we get $E\left(e_0\right)=E\left(e_1\right)=E\left(e_2\right)=0$, where $E\left(e_0^2\right)=\frac{(1-f)}{n}C_y^2$, $E\left(e_1^2\right)=\frac{(1-f)}{n}C_{x_1}^2$, $E\left(e_2^2\right)=\frac{(1-f)}{n}C_{x_2}^2$, $E\left(e_0e_1\right)=\frac{(1-f)}{n}\rho_{yx_1}C_yC_{x1}$,
$E\left(e_0e_2\right)=\frac{(1-f)}{n}\rho_{yx_2}C_yC_{x2}$ and $E\left(e_1e_2\right)=\frac{(1-f)}{n}\rho_{x_1x_2}C_{x1}C_{x2}$.

The proposed general class of estimators $\hat{\bar{Y}}_{gc}$ can be written terms of $e_0,\ e_1$ and e_2 as

$$\hat{\bar{Y}}_{gc}=K_1\bar{Y}(1+e_0)\left(\frac{T_1\bar{X}_1+P_1}{T_1\bar{X}_1\left(1+e_1\right)+P_1}\right)+K_2\bar{Y}(1+e_0)\left(\frac{T_2\bar{X}_1+P_2}{T_2\bar{X}_2\left(1+e_2\right)+P_2}\right)$$

$$\hat{\bar{Y}}_{gc}=K_1\bar{Y}(1+e_0)\left(\frac{1}{1+\frac{T_1\bar{X}_1e_1}{T_1\bar{X}_1+P_1}}\right)+K_2\bar{Y}(1+e_0)\left(\frac{1}{1+\frac{T_2\bar{X}_2e_2}{T_2\bar{X}_1+P_2}}\right)$$

$$\hat{\bar{Y}}_{gc}=K_1\bar{Y}(1+e_0)(1+\beta_1e_1)^{-1}+K_2\bar{Y}(1+e_0)(1+\beta_2e_2)^{-1} \tag{3.3}$$

Ignoring the higher order terms and also subtracting $\bar{Y}$ from both sides of equation (3.4), we get

$$\hat{\bar{Y}}_{gc}-\bar{Y}\cong\bar{Y}(e_0-K_1\beta_1e_1-K_2\beta_2e_2)$$

The MSE of the proposed class of estimators are obtained as follows:

$$\begin{aligned}MSE\left(\hat{\bar{Y}}_{gc}\right)&=E(\hat{\bar{Y}}_{gc}-\bar{Y})^2\\ &\cong\ \bar{Y}^2(E\left(e_0^2\right)+K_1^2\beta_1^2E\left(e_1^2\right)+K_2^2\beta_2^2E\left(e_2^2\right)-2K_1\beta_1E\left(e_0e_1\right)\\ &\qquad -2K_2\beta_2E\left(e_0e_2\right)+2K_1K_2\beta_1\beta_2E\left(e_1e_2\right))\end{aligned}$$

So,

$$\begin{gathered}MSE\left(\hat{\bar{Y}}_{gc}\right)\cong\frac{1-f}{n}\bar{Y}^2(C_y^2+K_1^2\beta_1^2C_{x_1}^2+K_2^2\beta_2^2C_{x_2}^2\\ -2K_1\beta_1\rho_{yx_1}C_yC_{x1}-2K_2\beta_2\rho_{yx_2}C_yC_{x2}+2K_1K_2\beta_1\beta_2\rho_{x_1x_2}C_{x1}C_{x2})\end{gathered} \tag{3.4}$$

where, $\beta_1=(\ \frac{T_1\bar{X}_1}{T_1\bar{x}_1+P_1})$, $\beta_2=(\ \frac{T_2\bar{X}_2}{T_2\bar{x}_2+P_2})$

The optimum values of K_1 and K_2 to minimize (3.5) for general class of estimators can easily be found as follows:

$$K_1^*=\frac{\beta_2^2C_{x_2}^2+\beta_1\rho_{yx_1}C_yC_{x1}-\beta_1\beta_2\rho_{x_1x_2}C_{x1}C_{x2}-\beta_2\rho_{yx_2}C_yC_{x2}}{\beta_1^2C_{x_1}^2-2\beta_1\beta_2\rho_{x_1x_2}C_{x1}C_{x2}+\beta_2^2C_{x_2}^2},$$

$$K_2^*=1-K_1^*.$$

So, the minimum MSE of general class of estimators are given by

$$MSE_{min}\left(\hat{\bar{Y}}_{gc}\right) \cong \frac{1-f}{n}\bar{Y}^2(C_y^2 + K_1^{*2}\beta_1^2 C_{x_1}^2 + K_2^{*2}\beta_2^2 C_{x_2}^2 \tag{3.5}$$
$$-2K_1^*\beta_1\rho_{yx_1}C_yC_{x1} - 2K_2^*\beta_2\rho_{yx_2}C_yC_{x2} + 2K_1^*K_2^*\beta_1\beta_2\rho_{x_1x_2}C_{x1}C_{x2})$$

4. Efficiency comparisons

In this section, the condition for which the proposed ratio estimators will have minimum mean square error compared to usual ratio estimator and existing ratio estimator for estimating the finite population mean have been derived algebraically.

4.1. Comparison with traditional ratio estimator. We compare the MSE of the proposed ratio estimator given in equation (3.2) with the MSE of the classical ratio estimator given in equation (2.1) as follows:

$$MSE_{min}\left(\hat{\bar{Y}}_{pj}\right) < MSE_{min}\left(\hat{\bar{Y}}_{MR}\right)$$

$$\Leftrightarrow (k_1^{*2}R_{p1j}^2 - \gamma_1^{*2})C_{x_1}^2 + (k_2^{*2}R_{p2j}^2 - \gamma_2^{*2})C_{x_2}^2$$
$$-2(k_1^*R_{p1j} - \gamma_1^*)\rho_{yx_1}C_yC_{x1} - 2(k_2^*R_{p2j} - \gamma_2^*)\rho_{yx_2}C_yC_{x2}$$
$$+2(k_1^*k_2^*R_{p1j}R_{p2j} - \gamma_1^*\gamma_2^*)\rho_{x_1x_2}C_{x1}C_{x2} < 0$$

where, $j = 1, 2, \ldots, 16$.

4.2. Comparison with Singh [23] ratio estimator. The proposed ratio estimator $\hat{\bar{Y}}_{pj}$ will be more efficient than that of [23] ratio estimator i.e. $\hat{\bar{Y}}_S$ if

$$MSE_{min}\left(\hat{\bar{Y}}_{pj}\right) < MSE\left(\hat{\bar{Y}}_S\right)$$

$$\Leftrightarrow (k_1^{*2}R_{p1j}^2 - 1)C_{x_1}^2 + (k_2^{*2}R_{p2j}^2 - 1)C_{x_2}^2 - 2(k_1^*R_{p1j} - 1)\rho_{yx_1}C_yC_{x1}$$
$$+2(k_1^*k_2^*R_{p1j}R_{p2j} - 1)\rho_{x_1x_2}C_{x1}C_{x2} - 2k_2^*R_{p2j}$$
$$\rho_{yx_2}C_yC_{x2} < 0$$

where $j = 1, 2, \ldots, 16$.

4.3. Comparison with Singh and Tailor [25] ratio estimator. The proposed ratio estimator $\hat{\bar{Y}}_{pj}$ will be more efficient than the Singh and Tailor [25] ratio estimator i.e. $\hat{\bar{Y}}_{ST}$ if

$$MSE_{min}\left(\hat{\bar{Y}}_{pj}\right) < MSE\left(\hat{\bar{Y}}_{ST}\right)$$

$$\Leftrightarrow (k_1^{*2}R_{p1j}^2 - \delta_1^*(\delta_1^* - 2k_{yx1}))C_{x_1}^2$$
$$+(k_2^{*2}R_{p2j}^2 - \delta_2^*\left(\delta_2^* + 2\left(k_{yx2} - \delta_1^*k_{x1x2}\right)\right)C_{x_2}^2$$
$$-2(k_1^*R_{p1j}\rho_{yx_1}C_{x1} + k_2^*R_{p2j}\rho_{yx_2}C_{x2})C_y$$
$$+2k_1^*k_2^*R_{p1j}R_{p2j}\rho_{x_1x_2}C_{x1}C_{x2} < 0$$

where $j = 1, 2, \ldots, 16$.

4.4. Comparison with Lu and Yan [16] ratio estimators. We compare the MSE of the proposed ratio estimator given in equation (3.4) with the MSE of the ratio estimator proposed by [16] given in equation (2.4) as follows:

$$
\begin{aligned}
&MSE_{min}\left(\hat{\hat{Y}}_{pj}\right) < MSE_{min}\left(\hat{\hat{Y}}_{i}\right)\\
&\qquad \Leftrightarrow (k_1^{*2} R_{p1j}^2 - a_1^{*2} R_{1i}^2) C_{x_1}^2 + (k_2^{*2} R_{p2j}^2 - a_2^{*2} R_{2i}^2) C_{x_2}^2\\
&-2(k_1^* R_{p1j} - a_1^* R_{1i}) \rho_{yx_1} C_y C_{x1} - 2(k_2^* R_{p2j} - a_2^* R_{2i}) \rho_{yx_2} C_y C_{x2}\\
&\qquad +2(k_1^* k_2^* R_{p1j} R_{p2j} - a_1^* a_2^* R_{1i} R_{2i}) \rho_{x_1x_2} C_{x1} C_{x2} < 0
\end{aligned}
$$

where $j = 1, 2, \ldots, 16$ and $i = 1, 2, \ldots, 9$. If the above condition is satisfied, then the proposed estimator $\hat{\hat{Y}}_{pj}$ will be more efficient than the $\hat{\hat{Y}}_{i}$ ratio estimator.

Table 1. The suitable choices of constant T_1, T_2, P_1, and P_2 for existing and proposed estimators.

Estimator	T_1	P_1	T_2	P_2
$\hat{\hat{Y}}_1$	1	C_{x1}	1	C_{x2}
$\hat{\hat{Y}}_2$	1	$\beta_{2(x_1)}$	1	$\beta_{2(x_2)}$
$\hat{\hat{Y}}_3$	$\beta_{2(x_1)}$	C_{x1}	$\beta_{2(x_2)}$	C_{x2}
$\hat{\hat{Y}}_4$	C_{x1}	$\beta_{2(x_1)}$	C_{x2}	$\beta_{2(x_2)}$
$\hat{\hat{Y}}_5$	1	ρ_{yx_1}	1	ρ_{yx_2}
$\hat{\hat{Y}}_6$	C_{x1}	ρ_{yx_1}	C_{x2}	ρ_{yx_2}
$\hat{\hat{Y}}_7$	ρ_{yx_1}	C_{x1}	ρ_{yx_2}	C_{x2}
$\hat{\hat{Y}}_8$	$\beta_{2(x_1)}$	ρ_{yx_1}	$\beta_{2(x_2)}$	ρ_{yx_2}
$\hat{\hat{Y}}_9$	ρ_{yx_1}	$\beta_{2(x_1)}$	ρ_{yx_2}	$\beta_{2(x_2)}$
$\hat{\hat{Y}}_{p1}$	1	MR_1	1	MR_2
$\hat{\hat{Y}}_{p2}$	1	TM_1	1	TM_2
$\hat{\hat{Y}}_{p3}$	1	HL_1	1	HL_2
$\hat{\hat{Y}}_{p4}$	1	DM_1	1	DM_2
$\hat{\hat{Y}}_{p5}$	$\beta_{2(x_1)}$	MR_1	$\beta_{2(x_2)}$	MR_2
$\hat{\hat{Y}}_{p6}$	$\beta_{2(x_1)}$	TM_1	$\beta_{2(x_2)}$	TM_2
$\hat{\hat{Y}}_{p7}$	$\beta_{2(x_1)}$	HL_1	$\beta_{2(x_2)}$	HL_2
$\hat{\hat{Y}}_{p8}$	$\beta_{2(x_1)}$	DM_1	$\beta_{2(x_2)}$	DM_2
$\hat{\hat{Y}}_{p9}$	C_{x1}	MR_1	C_{x2}	MR_2
$\hat{\hat{Y}}_{p10}$	C_{x1}	TM_1	C_{x2}	TM_2
$\hat{\hat{Y}}_{p11}$	C_{x1}	HL_1	C_{x2}	HL_2
$\hat{\hat{Y}}_{p12}$	C_{x1}	DM_1	C_{x2}	DM_2
$\hat{\hat{Y}}_{p13}$	ρ_{yx_1}	MR_1	ρ_{yx_2}	MR_2
$\hat{\hat{Y}}_{p14}$	ρ_{yx_1}	TM_1	ρ_{yx_2}	TM_2
$\hat{\hat{Y}}_{p15}$	ρ_{yx_1}	HL_1	ρ_{yx_2}	HL_2
$\hat{\hat{Y}}_{p16}$	ρ_{yx_1}	DM_1	ρ_{yx_2}	DM_2

Table 2. The values of constants and MSEs of the existing ratio estimators.

Estimator	Population 1			Population 1		
	R_{1i}	R_{2i}	MSE	R_{1i}	R_{2i}	MSE
$\hat{\hat{Y}}_{MR}$	–	–	10580.57	–	–	67682.99
$\hat{\hat{Y}}_{S}$	–	–	35056.92	–	–	2397933.00
$\hat{\hat{Y}}_{ST}$	–	–	10076.26	–	–	926358.10
$\hat{\hat{Y}}_{1}$	0.9966	0.9963	10555.66	0.9967	0.9993	68397.07
$\hat{\hat{Y}}_{2}$	0.9863	0.9816	10480.16	0.9976	0.9991	67967.88
$\hat{\hat{Y}}_{3}$	0.9988	0.9990	10572.04	0.9953	0.9994	68885.84
$\hat{\hat{Y}}_{4}$	0.9810	0.9757	10442.89	0.9974	0.9988	67843.83
$\hat{\hat{Y}}_{5}$	0.9979	0.9977	10564.99	0.9968	0.9992	68267.22
$\hat{\hat{Y}}_{6}$	0.9970	0.9970	10559.00	0.9966	0.9989	68173.40
$\hat{\hat{Y}}_{7}$	0.9924	0.9917	10525.67	0.9964	0.9993	68474.96
$\hat{\hat{Y}}_{8}$	0.9993	0.9994	10575.24	0.9954	0.9992	68742.67
$\hat{\hat{Y}}_{9}$	0.9700	0.9601	10362.63	0.9973	0.9990	68009.70

Table 3. The values of constants and MSEs of the existing ratio estimators.

Estimator	Population 1			Population 1		
	R_{p1j}	R_{p2j}	MSE	R_{p1j}	R_{p2j}	MSE
$\hat{\hat{Y}}_{p1}$	0.4234	0.3840	9007.99	0.3323	0.3855	16197.60
$\hat{\hat{Y}}_{p2}$	0.5628	0.5464	8887.83	0.5794	0.5474	13504.57
$\hat{\hat{Y}}_{p3}$	0.5237	0.5201	8884.53	0.5338	0.5198	13344.46
$\hat{\hat{Y}}_{p4}$	0.4831	0.4908	8899.02	0.5080	0.4947	13337.08
$\hat{\hat{Y}}_{p5}$	0.6812	0.6992	9064.65	0.2578	0.3971	13388.00
$\hat{\hat{Y}}_{p6}$	0.7893	0.8180	9401.53	0.4902	0.5594	16973.96
$\hat{\hat{Y}}_{p7}$	0.7619	0.8017	9297.10	0.4442	0.5320	16476.45
$\hat{\hat{Y}}_{p8}$	0.7312	0.7824	9195.52	0.4188	0.5069	14959.01
$\hat{\hat{Y}}_{p9}$	0.3458	0.3185	9193.93	0.3205	0.3203	20371.89
$\hat{\hat{Y}}_{p10}$	0.4810	0.4746	8906.98	0.5664	0.4759	13709.12
$\hat{\hat{Y}}_{p11}$	0.4418	0.4484	8940.52	0.5205	0.4484	14124.45
$\hat{\hat{Y}}_{p12}$	0.4023	0.4196	8985.58	0.4946	0.4237	14676.30
$\hat{\hat{Y}}_{p13}$	0.2483	0.2190	9496.34	0.3129	0.3712	17168.84
$\hat{\hat{Y}}_{p14}$	0.3668	0.3515	9124.94	0.5577	0.5322	13389.48
$\hat{\hat{Y}}_{p15}$	0.3310	0.3278	9207.19	0.5117	0.5046	13310.59
$\hat{\hat{Y}}_{p16}$	0.2961	0.3025	9283.07	0.4858	0.4795	13427.61

5. Empirical Study

The performance of the proposed ratio estimators and the existing ratio estimators is evaluated by using two natural populations. The population 1 is taken from Singh and Chaudhary [22] page 177 and population 2 is taken from Murthy [18] page 228. The characteristics of the two populations are given below:

Population 1 (Singh and Chaudhary [22])

Y=Area under Wheat in 1974

X_1=Area under Wheat in 1971

X_2=Area under Wheat in 1973

$N = 34,\ n = 20,\ \bar{Y} = 856.412,\ \bar{X}_1 = 208.882,\ \bar{X}_2 = 199.441, C_y = 0.8561,$

$C_{x_1} = 0.721,\ C_{x_2} = 0.753, \rho_{yx_1} = 0.449,\ \rho_{yx_2}=0.445,\ \rho_{x_1x_2} = 0.980,$

$\beta_{2(x_1)} = 2.910,\ \beta_{2(x_2)} = 3.732,\ MR_1 = 284.500,\ MR_2 = 320.000,$

$TM_1 = 162.250,\ TM_2 = 165.562,\ HL_1 = 190.000,\ HL_2 = 184.000,$

$DM_1 = 223.467,\ DM_2 = 206.944.$

Population 2 (Murthy [18])

Y=Output

X_1=Number of workers

X_2=Fixed capital

$N = 80,\ n = 20,\ \bar{Y} = 5182.637,\ \bar{X}_1 = 285.125,\ \bar{X}_2 = 1126.463, C_y = 0.354,$

$C_{x_1} = 0.948,\ C_{x_2} = 0.751, \rho_{yx_1} = 0.915,\ \rho_{yx_2}=0.941,\ \rho_{x_1x_2} = 0.988,$

$\beta_{2(x_1)} = 0.698,\ \beta_{2(x_2)} = 1.050,\ MR_1 = 573.000,\ MR_2 = 1795.500,$

$TM_1 = 206.937,\ TM_2 = 931.562,\ HL_1 = 249.000,\ HL_2 = 1040.500,$

$DM_1 = 276.189,\ DM_2 = 1150.700.$

The values of constants and the MSE of the existing and proposed ratio estimators using the information of two auxiliary variables are given in Tables 2 and 3, respectively. It can be observed that the constants and the MSE of the suggested ratio estimators are smaller than the usual ratio estimator and the existing ratio estimators consider in this study (cf. Tables 2-3). From Table 3, it is evident that the proposed estimators perform better than the usual ratio estimator and the existing ratio estimators in terms of MSE, which shows that the proposed estimators are more efficient.

The comparison of the proposed ratio estimators with the traditional ratio and the existing ratio estimators are also shown by graphically for all the populations considered in this study. From Figures 1-2, it can be seen that the proposed estimators have smaller values of MSE as compared to the usual ratio estimator and the existing ratio estimators, which indicates that the performance of the proposed estimators are better as compared to the traditional ratio estimator, Singh [23] estimator, Singh and Tailor [25] estimator and Lu and Yan [16] estimators.

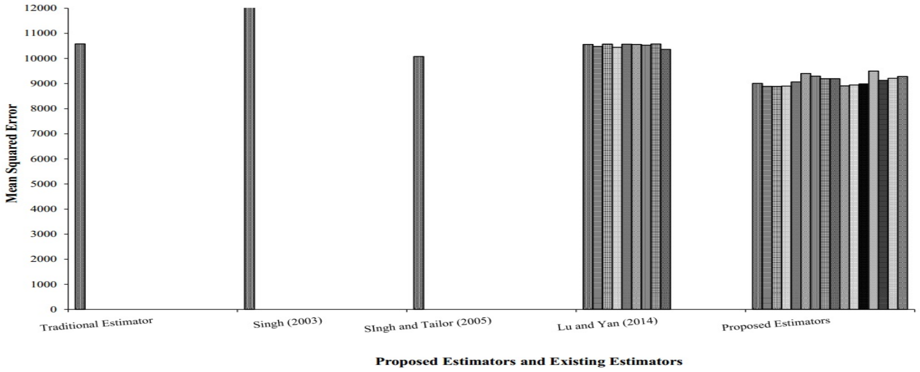

Figure 1. Mean squared error of the proposed and existing estimators of population 1.

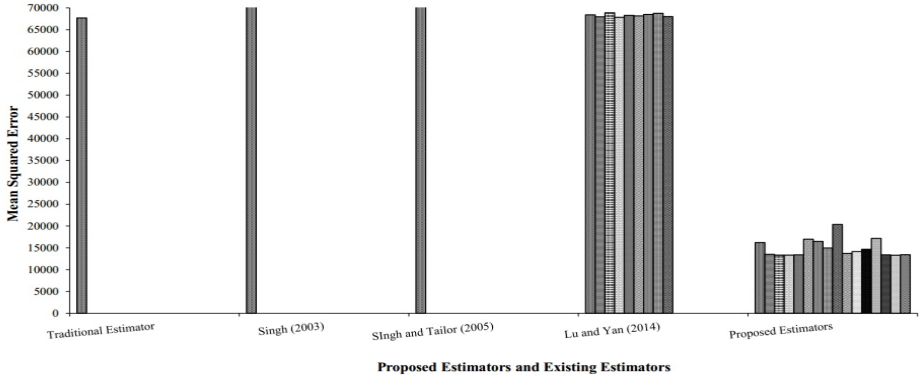

Figure 2. Mean squared error of the proposed and existing estimators of population 2.

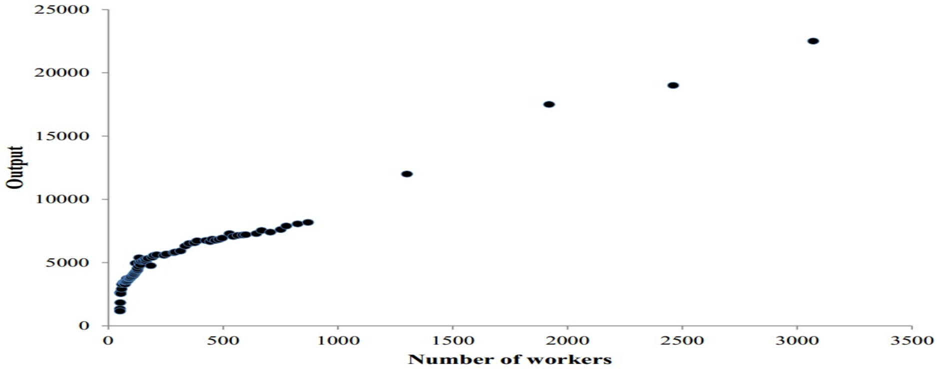

Figure 3. Scatter graph of first auxiliary and study variables.

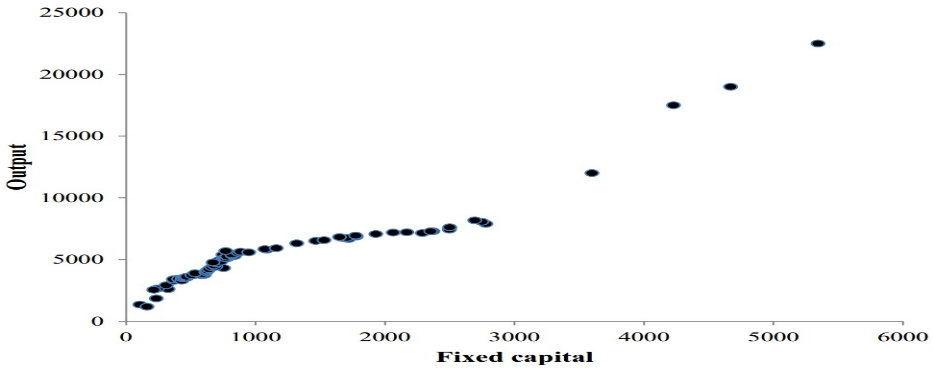

Figure 4. Scatter graph of second auxiliary and study variables.

Table 4. The values MSEs of the existing and proposed estimators for outliers data.

Existing	MSE	Proposed	MSE
$\hat{\bar{Y}}_{MR}$	151170.40	$\hat{\bar{Y}}_{p1}$	23377.04
$\hat{\bar{Y}}_{S}$	2928277.00	$\hat{\bar{Y}}_{p2}$	18051.03
$\hat{\bar{Y}}_{ST}$	1121996.00	$\hat{\bar{Y}}_{p3}$	19849.40
$\hat{\bar{Y}}_{1}$	152013.10	$\hat{\bar{Y}}_{p4}$	18406.64
$\hat{\bar{Y}}_{2}$	151555.80	$\hat{\bar{Y}}_{p5}$	25556.96
$\hat{\bar{Y}}_{3}$	156380.50	$\hat{\bar{Y}}_{p6}$	24281.94
$\hat{\bar{Y}}_{4}$	151367.60	$\hat{\bar{Y}}_{p7}$	21726.77
$\hat{\bar{Y}}_{5}$	151824.10	$\hat{\bar{Y}}_{p8}$	19016.20
$\hat{\bar{Y}}_{6}$	151648.40	$\hat{\bar{Y}}_{p9}$	25035.24
$\hat{\bar{Y}}_{7}$	152047.30	$\hat{\bar{Y}}_{p10}$	19106.17
$\hat{\bar{Y}}_{8}$	152160.60	$\hat{\bar{Y}}_{p11}$	18584.91
$\hat{\bar{Y}}_{9}$	151562.80	$\hat{\bar{Y}}_{p12}$	18698.78
–	–	$\hat{\bar{Y}}_{p13}$	24389.50
–	–	$\hat{\bar{Y}}_{p14}$	20413.35
–	–	$\hat{\bar{Y}}_{p15}$	19353.86
–	–	$\hat{\bar{Y}}_{p16}$	18692.03

5.1. Robustness of the proposed estimators.

As in the earlier sections, it is mentioned that measures used in this study such as tri-mean, mid-range, Hodges-Lehmann and decile mean are robust against outliers. Thus when there is a outlier in the data these measures are perform efficiently as compared to other measures of locations. So, in this section, we check the efficiency of our proposed estimators in case of outliers. For this purpose, we considered the data of Population 2 and introduced some outliers in this data. From Figures 3 and 4, we clearly see that there are outliers in the data, so we can except the proposed estimators to perform better than the usual and existing estimators consider in this study.

We obtain the MSE values of the existing and proposed estimators as defined in Sections 2 and 3, respectively by using outliers data. The MSE values of the existing and proposed estimators are given in Table 4. From Table 4, it is observed that the proposed estimators have smaller values of MSE as compared to the usual ratio estimator and existing estimators, which indicates that the proposed estimators are more efficient in the presence of outliers.

To show the dominance of the proposed ratio estimators over the existing estimators, we have also found the percent relative efficiencies (PREs) for population 2 in case of excluded and included outliers in the data. The PREs of the proposed estimators (p) with respect to the existing estimators (e) is computed as

$$PRE(e,p) = \frac{MSE(e)}{MSE(p)} * 100 \tag{5.1}$$

and are given in Tables 5 and 6, respectively (see Appendix).

If the percentage relative efficiency value found from (5.1) is greater than 100, then it is seeming that the proposed estimators are more efficient as compared to the usual ratio estimator and existing estimators. Therefore, from Tables 5 and 6, we see that all the proposed estimators are more efficient than the traditional ratio estimator and existing estimators consider in this study. It is to be also noted that the values of relative efficiencies of the proposed estimators with respect to the existing estimators in Table 6 would increase dramatically, which shows that the efficiencies of the proposed estimators would increase significantly, if there were more outliers in the data.

6. Summary and conclusions

The study has proposed a variety of two auxiliary based ratio estimators using tri-mean, mid-range, Hodges-Lehmann, decile mean, coefficient of variation, coefficient of kurtosis and correlation coefficient. It is observed that the proposed estimators outperform the usual ratio estimator and the existing ratio estimators in terms of mean squared error under all the populations considered for the numerical study. Moreover, robustness to extreme observations is an added feature of the proposed estimators. Hence, we recommended the use of the proposed ratio estimators over the usual and other existing ratio estimators, especially in the presence of unusual observations in the data.

Appendix

Table 5. Percentage Relative Efficiency of existing estimators with respect to proposed estimators of Population 2 without outlier data.

Proposed Estimators	**Existing Estimators**											
	$\hat{Y}_{MR}$	$\hat{Y}_{S}$	$\hat{Y}_{ST}$	$\hat{Y}_{1}$	$\hat{Y}_{2}$	$\hat{Y}_{3}$	$\hat{Y}_{4}$	$\hat{Y}_{5}$	$\hat{Y}_{6}$	$\hat{Y}_{7}$	$\hat{Y}_{8}$	$\hat{Y}_{9}$
$\hat{Y}_{p1}$	417.9	14804.2	5719.1	422.3	419.6	425.3	418.9	421.5	420.9	422.7	424.4	419.9
$\hat{Y}_{p2}$	501.2	17756.5	6859.6	506.5	503.3	510.1	502.4	505.5	504.8	507.1	509.0	503.6
$\hat{Y}_{p3}$	507.2	17969.5	6941.9	512.6	509.3	516.2	508.4	511.6	510.9	513.1	515.1	509.6
$\hat{Y}_{p4}$	507.5	17979.4	6945.7	512.8	509.6	516.5	508.7	511.9	511.2	513.4	515.4	509.9
$\hat{Y}_{p5}$	505.5	17911.1	6919.3	510.9	507.7	514.5	506.8	509.9	509.2	511.5	513.5	508.0
$\hat{Y}_{p6}$	398.7	14127.1	5457.5	403.0	400.4	405.8	399.7	402.2	401.6	403.4	405.0	400.7
$\hat{Y}_{p7}$	410.8	14553.7	5622.3	415.1	412.5	418.1	411.8	414.3	413.8	415.6	417.2	412.8
$\hat{Y}_{p8}$	452.5	16030.0	6192.6	457.2	454.4	460.5	453.5	456.4	455.7	457.8	459.5	454.6
$\hat{Y}_{p9}$	332.2	11770.8	4547.2	335.7	333.6	338.1	333.0	335.1	334.6	336.1	337.4	333.8
$\hat{Y}_{p10}$	493.7	17491.5	6757.2	498.9	495.8	502.5	494.9	498.0	497.3	499.5	501.4	496.1
$\hat{Y}_{p11}$	479.2	16977.2	6558.5	484.2	481.2	487.7	480.3	483.3	482.7	484.8	486.7	481.5
$\hat{Y}_{p12}$	461.2	16338.8	6311.9	466.0	463.1	469.4	462.3	465.2	464.5	466.6	468.4	463.4
$\hat{Y}_{p13}$	394.2	13966.8	5395.6	398.4	395.9	401.2	395.2	397.6	397.1	398.8	400.4	396.1
$\hat{Y}_{p14}$	505.5	17909.1	6918.6	510.8	507.6	514.5	506.7	509.9	509.2	511.4	513.4	507.9
$\hat{Y}_{p15}$	508.5	18015.2	6959.6	513.9	510.6	517.5	509.7	512.9	512.2	514.4	516.5	510.9
$\hat{Y}_{p16}$	504.1	17858.2	6898.9	509.4	506.2	513.0	505.3	508.4	507.7	510.0	512.0	506.5

Table 6. Percentage Relative Efficiency of existing estimators with respect to proposed estimators of Population 2 with outlier data.

Proposed Estimators	**Existing Estimators**											
	$\hat{Y}_{MR}$	$\hat{Y}_{S}$	$\hat{Y}_{ST}$	$\hat{Y}_{1}$	$\hat{Y}_{2}$	$\hat{Y}_{3}$	$\hat{Y}_{4}$	$\hat{Y}_{5}$	$\hat{Y}_{6}$	$\hat{Y}_{7}$	$\hat{Y}_{8}$	$\hat{Y}_{9}$
$\hat{Y}_{p1}$	646.7	12526.3	4799.6	650.3	648.3	668.9	647.5	649.5	648.7	650.4	650.9	648.3
$\hat{Y}_{p2}$	837.5	16222.2	6215.7	842.1	839.6	866.3	838.6	841.1	840.1	842.3	842.9	839.6
$\hat{Y}_{p3}$	761.6	14752.5	5652.5	765.8	763.5	787.8	762.6	764.9	764.0	766.0	766.6	763.6
$\hat{Y}_{p4}$	821.3	15908.8	6095.6	825.9	823.4	849.6	822.4	824.8	823.9	826.0	826.7	823.4
$\hat{Y}_{p5}$	591.5	11457.8	4390.2	594.8	593.0	611.9	592.3	594.1	593.4	594.9	595.4	593.0
$\hat{Y}_{p6}$	622.6	12059.5	4620.7	626.0	624.2	644.0	623.4	625.3	624.5	626.2	626.6	624.2
$\hat{Y}_{p7}$	695.8	13477.7	5164.1	699.7	697.6	719.8	696.7	698.8	698.0	699.8	700.3	697.6
$\hat{Y}_{p8}$	795.0	15398.9	5900.2	799.4	797.0	822.4	796.0	798.4	797.5	799.6	800.2	797.0
$\hat{Y}_{p9}$	603.8	11696.6	4481.7	607.2	605.4	624.6	604.6	606.4	605.7	607.3	607.8	605.4
$\hat{Y}_{p10}$	791.2	15326.3	5872.4	795.6	793.2	818.5	792.2	794.6	793.7	795.8	796.4	793.3
$\hat{Y}_{p11}$	813.4	15756.2	6037.1	817.9	815.5	841.4	814.5	816.9	816.0	818.1	818.7	815.5
$\hat{Y}_{p12}$	808.5	15660.3	6000.4	813.0	810.5	836.3	809.5	811.9	811.0	813.1	813.7	810.5
$\hat{Y}_{p13}$	619.8	12006.3	4600.3	623.3	621.4	641.2	620.6	622.5	621.8	623.4	623.9	621.4
$\hat{Y}_{p14}$	740.5	14344.9	5496.4	744.7	742.4	766.1	741.5	743.7	742.9	744.8	745.4	742.5
$\hat{Y}_{p15}$	781.1	15130.2	5797.3	785.4	783.1	808.0	782.1	784.5	783.6	785.6	786.2	783.1
$\hat{Y}_{p16}$	808.7	15665.9	6002.5	813.3	810.8	836.6	809.8	812.2	811.3	813.4	814.0	810.8

Acknowledgments The authors are grateful to the Editor-in-chief Prof. Dr. Cem Kadilar and referees for their constructive comments that led to substantial improvements in the article. The third author is indebted to the King Fahd University of Petroleum and Minerals (KFUPM), Dhahran, Saudi Arabia, for providing excellent research facilities.

References

[1] Abid, M., Abbas, N., Nazir, H. Z. and Lin, Z. *Enhancing the mean ratio estimators for estimating population mean using non-conventional location parameters*, Revista Colombiana de Estadistica **39** (1), 63-79, 2016(a).

[2] Abid, M., Abbas, N. and Riaz, M. *Improved modified ratio estimators of population mean based on deciles*, Chiang Mai Journal of Science **43** (1), 1311-1323, 2016(b).

[3] Abid, M., Abbas, N., Sherwani, R. A. K. and Nazir, H. Z. *Improved ratio estimators for the population mean using non-conventional measures of dispersion*, Pakistan Journal of Statistics and Operation Research **12** (2), 353-367, 2016(c).

[4] Abid, M., Sherwani, R. A. K., Abbas, N. and Ali, M. R. Some *Improved modified ratio estimators based on dedile mean of an auxiliary variable*, Pakistan Journal of Statistics and Operation Research **12** (4), 787-797, 2016.

[5] Abu-Dayeh, W.A., Ahmed, M.S., Ahmed, R.A. and Muttlak, H.A. *Some estimators of a finite population mean using auxiliary information*, Applied Mathematics and Computation 139, 287–298, 2003.

[6] Al-Hossain, A.Y., and Khan, M. *Efficiency of ratio, product and regression estimators under maximum and minimum values, using two auxiliary variables*, Journal of Applies Mathematics**201**, 1-6, 2014.

[7] Cochran, W. G. *Sampling techniques* (John Wiley and Sons, New-York,1977) .

[8] Ferrell, E.B. *Control charts using midranges and medians*, Industrial Quality Control **9**, 30–34, 1953.

[9] Hettmansperger, T.P., and McKean, J. W. *Robust nonparametric statistical methods* (Arnold London, 1998).

[10] Kadilar, C., and Cingi, H. *Ratio estimators in simple random sampling*, Applied Mathematics and Computation **151**, 893-902, 2004.

[11] Kadilar, C., and Cingi, H. *Estimator of a population mean using two auxiliary variables in simple random sampling*, International Mathematical Journal **5**,, 357-360, 2004.

[12] Kadilar, C., and Cingi, H. *A new estimator using two auxiliary variables*, Applied Mathematics and Computation **162**, 901-908, 2005.

[13] Kadilar, C., and Cingi, H. *An Improvement in estimating the population mean by using the correlation coefficient*, Hacettepe Journal of Mathematics and Statistics**55** (1), 103–109, 2006.

[14] Khare, B. B., Srivastava, U., and Kumar, K. *A generalized chain ratio in regression estimator for population mean using two auxiliary characters in sample survey*, Journal of Scientific Research **57**, 147-153, 2013.

[15] Lu, J. *The chain ratio estimator and regression estimator with linear combination of two auxiliary variables*, PLOS ONE **11** (8) , 1-4, 2013.

[16] Lu, J., and Yan, Z. *A class of ratio estimators of a finite population mean using two auxiliary variables*, PLOS ONE **2** (9), 1-6, 2014.

[17] Lu, J., Yan, Z., and Peng X. *A new exponential ratio type estimator with linear combination of two auxiliary variables*, PLOS ONE, **12** (9), 1-10, 2014.

[18] Murthy, M. N. *Sampling Theory and Methods* (Statistical Publishing Society, Calcutta, India, 1967.)

[19] Olkin, I. *Multivariate ratio estimation for finite populations*, Biometrika **45**, 154-165, 1958.

[20] Perri, P. F. *Improved ratio-cum-product type estimators*, Statistics in Transition **1** (8), 51-69, 2007.

[21] Raj, D. On *a method of using multi-auxiliary information in sample surveys*, Journal of the American Statistical Association **60**, 154-165, 1965.

[22] Singh, D., and Chaudhary, F. S. *Theory and analysis of sample survey designs* (New Age International Publisher, New Delhi, India, 1986).

[23] Singh, G. N. On *the improvement of product method of estimation in sample surveys*, Journal of the Indian Society of Agricultural Statistics **56** (3), 267-265, 2003.

[24] Singh, H. P., and Tailor, R. *Use of known correlation coefficient in estimating the finite population means*, Statistics in Transition **6** (4), 555-560, 2003.

[25] Singh, H. P., and Tailor, R. *Estimation of finite population mean using known correlation coefficient between auxiliary characters*, Statistica **65**, 407-418, 2005

[26] Sisodia, B. V. S., and Dwivedi, V. K. *A modified ratio estimator using coefficient of variation of auxiliary variable*, *Journal* of the Indian Society of Agricultural Statistics **33** (1), 13-18, 1981.

[27] Srivastava, S.K. *A generalized estimator for the mean of a finite population using multi-auxiliary information*, Journal of American Statistical Association **66**, 404-407, 1971.

[28] Srivastava, S.K., and Jhajj, H.S. *A class of estimators of the population mean using multi-auxiliary information*, Calcutta Statistical Association Bulletin **32**, 47-56, 1983.

[29] Subramani, J., and Kumarapandiyan, G. *Estimation of population mean using known median and co-efficient of skewness*, American Journal of Mathematics and Statistics **2** (5), 101-107, 2012.

[30] Subramani, J., and Kumarapandiyan, G. *Estimation of population mean using co-efficient of variation and median of an auxiliary variable*, International Journal of Probability and Statistics**1** (4), 111-118, 2012.

[31] Subramani, J., and Kumarapandiyan, G. *Modified ratio estimators using known median and co-efficient of kurtosis*, American Journal of Mathematics and Statistics**2** (4), 95-100, 2012.

[32] Subramani, J., and Kumarapandiyan, G. *A class of modified ratio estimators using deciles of an auxiliary variable*, International Journal of Statistical Applications **2**, 101-107, 2012.

[33] Subramani, J., and Kumarapandiyan, G. *A new modified ratio estimator for estimation of population mean when median of the auxiliary variable is known*, Pakistan Journal of Statistics and Operation Research**9** (2), 137-145, 2013.

[34] Subramani, J., and Prabavathy, G. *Two parameters modified ratio estimators with two auxiliary variables for estimation of finite population mean with known skewness, kurtosis and correlation coefficient*, Journal of Modern Applied Statistical Methods**13** (1), 199-222, 2014.

[35] Upadhyaya, L. N., and Singh, H. P. *Use of transformed auxiliary variable in estimating the finite population mean*, Biometrical Journal **41** (5), 627-636, 1999.

[36] Wang, T., Li, Y., and Cui, H. *On weighted randomly trimmed means*, Journal of Systems Science and Complexity **20**, 47–65, 2007.

[37] Yan, Z., and Tian, B. *Ratio method to the mean estimation using coefficient of skewness of auxiliary variable*, ICICA 2010, Part II, CCIS**106**, 103-110, 2010.