Some Notes On (2,0)-Semitensor Bundle

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Abstract

We investigate some lifts of vector fields on a cross-section in the semi-tensor (pull-back) bundle tM of tensor bundle of type (2,0) by using projection (submersion) of the tangent bundle TM and we find some relation for them.

Keywords: Cross-section, pull-back bundle, tangent bundle, semi-tensor bundle.

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1. Introduction

Let $M_n$ be an n-dimensional differentiable manifold of class $C^\infty$, and let $(T(M_n), \pi_1, M_n)$ be a tangent bundle over $M_n$. We use the notation $(x') = (x^\alpha, x^\beta)$, where the indices $i, j, \ldots$ run from 1 to 2$n$, the indices $\alpha, \beta, \ldots$ from 1 to $n$ and the indices $\alpha, \beta, \ldots$ from $n + 1$ to 2$n$, $x^\alpha$ are coordinates in $M_n$, $x^\alpha = y^\beta$ are fibre coordinates of the tangent bundle $T(M_n)$.

Let now $(\tilde{T}^2_0(M_n), \tilde{x}, \tilde{M}_n)$ be a tensor bundle of the type $(2,0)$ ([4], [7], [9], p.118), [11]) over base space $M_n$, and let $T(M_n)$ be tangent bundle determined by a natural projection (submersion) $\pi_1 : T(M_n) \to M_n$. The semi-tensor bundle (pull-back) ([5],[6],[10],[12],[14],[15]) of the $(2,0)$-tensor bundle $(\tilde{T}^2_0(M_n), \tilde{x}, \tilde{M}_n)$ is the bundle $(\tilde{T}^2_0(M_n), \pi_2, T(M_n))$ over tangent bundle $T(M_n)$ with a total space

$$\tilde{T}^2_0(M_n) = \{ (\{x^\alpha, x^\beta\}, x^\beta) \in T(M_n) \times (\tilde{T}^2_0)_{\pi_1}(M_n) : \pi_1 (x^\alpha, x^\beta) = \tilde{x} (x^\alpha, x^\beta) = (x^\alpha) \}$$

and with the projection map $\pi_2 : \tilde{T}^2_0(M_n) \to T(M_n)$ defined by $\pi_2 (x^\alpha, x^\beta, \tilde{x}) = (x^\alpha, x^\beta)$, where $(\tilde{T}^2_0)_{\pi_1}(M_n) (x = \pi_1 (\tilde{x}), \tilde{x} = (x^\alpha, x^\beta) \in T(M_n))$ is the tensor space at a point $x$ of $M_n$, where $x^\alpha = \tilde{\beta}^{\alpha} (\tilde{x}, \tilde{y}, \ldots = 2n + 1, \ldots, 2n + n^2)$ are fiber coordinates of the tensor bundle $\tilde{T}^2_0(M_n)$.

The generalization of pull-back bundles to higher order cases is known as Pontryagin bundles [8].

If $(x') = (x^\alpha, x^\beta, x^\gamma)$ is another system of local adapted coordinates in the semi-tensor bundle $\tilde{T}^2_0(M_n)$, then we have

$$\begin{align*}
    x^\gamma &= \frac{\partial x^\gamma}{\partial x^\beta} y^\beta, \\
    x^\alpha &= x^\alpha (x^\beta), \\
    x^\beta &= x^\beta (x^\beta).
\end{align*}$$

(1.1)

The Jacobian of (1.1) has components

$$\tilde{A} = (A^A_\beta) = \begin{pmatrix}
    A^A_\beta & A^A_\beta x^\alpha \\
    0 & A^A_\beta \\
    0 & 0 & 0
\end{pmatrix}.$$

(1.2)

where $I = (\alpha, \beta, \gamma)$, $J = (\beta, \gamma, \lambda)$, $I, J, \ldots = 1, \ldots, 2n + n^2$, $A^A_\beta = \frac{\partial x^\gamma}{\partial x^\beta}$. It is easily verified that the condition $\text{Det} \tilde{A} \neq 0$ is equivalent to the condition:

$$\text{Det}(A^A_\beta) \neq 0, \text{Det}(A^A_\beta) \neq 0, \text{Det}(A^A_\beta A^A_\beta) \neq 0.$$
Also, \( \dim \tau^2_q(M_n) = 2n + n^2 \).

We note that cross-sections for \((2, 0)\)–tensor bundle and semi-tensor bundle of the type \((2, 0)\) were examined in \([2, 3]\). The main purpose of this paper is to study the behaviour of complete lifts of vector fields on cross-sections for \((2, 0)\)–semi tensor (pull-back) bundle by using projection of the tangent bundle \(T(M_n)\). We denote by \(\mathfrak{Z}^0_q(T(M_n))\) and \(\mathfrak{Z}^0_q(M_n)\) the modules over \(F(T(M_n))\) and \(F(M_n)\) of all tensor fields of the type \((p, q)\) on \(T(M_n)\) and \(M_n\), respectively, where \(F(T(M_n))\) and \(F(M_n)\) denote the rings of real-valued \(C^\infty\)–functions on \(T(M_n)\) and \(M_n\), respectively.

2. Vertical lifts of tensor fields and \(\gamma\)–operator

Let \(A \in \mathfrak{Z}^0_q(T(M_n))\). On putting

\[
\nu^v_A = \begin{pmatrix} \nu^v_A^\alpha \\ \nu^v_A^\beta \\ \nu^v_A^\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A^{\alpha \alpha} \end{pmatrix},
\]

from (1.2), we easily see that with \(\nu^v A = \tilde{A}(\nu^v A)\). The vector field \(\nu^v A \in \mathfrak{Z}^1_q(\tau^2_q(M_n))\) is called the vertical lift of \(A \in \mathfrak{Z}^0_q(T(M_n))\) to the semi-tensor bundle of the type \((2, 0)\).

For any \(\varphi \in \mathfrak{Z}^1_q(M_n)\), if we take account of (1.2), we can prove that \((\gamma \varphi)' = \tilde{A}(\gamma \varphi)\). Where \(\gamma \varphi\) is a vector field in \(\pi^{-1}(U)\) defined by

\[
\gamma \varphi = (\gamma \varphi)' = \begin{pmatrix} 0 \\ 0 \\ \epsilon^\alpha \partial^\alpha \phi^\alpha + \epsilon^\alpha \epsilon^\beta \phi^\beta \end{pmatrix}.
\]

From (1.2) we easily see that the vector fields \(\gamma \varphi\) defined in each \(\pi^{-1}(U) \subset \tau^2_q(M_n)\) determine global vertical vector fields on \(\tau^2_q(M_n)\). We call \(\gamma \varphi\) the vertical-vector lift of the tensor field \(\varphi \in \mathfrak{Z}^1_q(M_n)\) to \(\tau^2_q(M_n)\).

For any \(\varphi \in \mathfrak{Z}^1_q(T(M_n))\), if we take account of (1.2), we can prove that \((\gamma \varphi)' = \tilde{A}(\gamma \varphi)\), where \(\gamma \varphi\) is a vector field defined by

\[
\gamma \varphi = \begin{pmatrix} \gamma^\varphi \phi^\beta \\ 0 \\ 0 \end{pmatrix}
\]

with respect to the coordinates \((\gamma^\varphi, \gamma^\varepsilon, \gamma^\gamma)\).

3. Complete lifts of vector fields

Let \(X \in \mathfrak{Z}^0_q(T(M_n))\), i.e. \(X = X^\alpha (x^\alpha) \partial_\alpha\). The complete lift \(c X\) of \(X\) to tangent bundle is defined by \(c X = X^\alpha \partial_\alpha + \gamma^\varepsilon \partial_\varepsilon X^\alpha \partial_\varepsilon\) \([13]\), p.15. On putting

\[
c^\gamma X = \begin{pmatrix} c^\gamma X^\beta \\ c^\gamma X^\varepsilon \\ c^\gamma X^\gamma \end{pmatrix} = \begin{pmatrix} \gamma^\varepsilon \partial_\varepsilon X^\beta \\ X^\beta \\ \epsilon^\alpha \epsilon^\beta \partial_\beta X^\alpha + \epsilon^\alpha \epsilon^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix},
\]

from (1.2), we easily see that \(c^\gamma X' = \tilde{A}(c^\gamma X)\). The vector field \(c^\gamma X\) is called the complete lift of \(c X \in \mathfrak{Z}^1_q(T(M_n))\) to \(\tau^2_q(M_n)\).

\textbf{Proof.} If \(X \in \mathfrak{Z}^1_q(M_n)\) and \(\begin{pmatrix} c^\gamma X^\beta \\ c^\gamma X^\varepsilon \\ c^\gamma X^\gamma \end{pmatrix}\) are components of \((c^\gamma X)'\) with respect to the coordinates \((\gamma^\varphi, \gamma^\varepsilon, \gamma^\gamma)\) on \(\tau^2_q(M_n)\), then we have by (1.2) and (3.1):

\[
(c^\gamma X)' = A^L_{\alpha^\varphi}(c^\gamma X)^\alpha + A^L_{\alpha^\varepsilon}(c^\gamma X)^\alpha + A^L_{\alpha^\gamma}(c^\gamma X)^\gamma.
\]

Firstly, if \(J = \gamma^\gamma\), we have

\[
(c^\gamma X)^\gamma = A^L_{\alpha^\varepsilon}(c^\gamma X)^\alpha + A^L_{\alpha^\gamma}(c^\gamma X)^\gamma = \epsilon^\alpha \epsilon^\beta \partial_\beta X^\alpha + \epsilon^\alpha \epsilon^\varepsilon \partial_\varepsilon X^\alpha = \gamma^\varepsilon \partial_\varepsilon (A^L_{\alpha^\gamma} X^\alpha)
\]

\(c^\gamma X' = \tilde{A}^L_{\alpha^\varepsilon} (c^\gamma X)^\alpha + A^L_{\gamma^\alpha}(c^\gamma X)^\gamma + \epsilon^\alpha \epsilon^\varepsilon \partial_\varepsilon X^\alpha\)

\(= \gamma^\varepsilon \partial_\varepsilon (A^L_{\alpha^\gamma} X^\alpha)\)

\(= \gamma^\varepsilon \partial_\varepsilon X^\beta\)
by virtue of (1.2) and (3.1). Secondly, if \( J = \beta \), we have
\[
\left( ^{(cc)}X \right)^{\beta} = A_{\beta\alpha}^{\beta}(ccX)^{\alpha} + A_{\beta\alpha}^{\beta}(ccX)^{\alpha} + A_{\beta\alpha}^{\beta}(ccX)^{\alpha}
\]
\[
\Rightarrow A_{\beta\alpha}^{\beta}X^{\alpha} = X^{\beta}
\]
by virtue of (1.2) and (3.1). Thirdly, if \( J = \overline{\beta} \), then we have
\[
\left( ^{(cc)}X \right)^{\overline{\beta}} = t^{\alpha\beta}(\partial_{\alpha}X^{\alpha})A_{\alpha\beta}^{\beta} + t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})X^{\alpha}
\]
\[
+ t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})X^{\alpha} + t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})X^{\alpha}
\]
\[
= \sum_{p=1}^{4} a_{p}
\]
where
\[
a_1 = t^{\alpha\beta}(\partial_{\alpha}X^{\alpha})A_{\alpha\beta}^{\beta}
\]
\[
a_2 = t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})X^{\alpha}
\]
\[
a_3 = t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})X^{\alpha}
\]
\[
a_4 = t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})X^{\alpha}
\]
On the other hand,
\[
A_{\alpha}^{\beta}(^{(cc)}X)^{\alpha} = A_{\alpha}^{\beta}(^{(cc)}X)^{\alpha} + A_{\alpha}^{\beta}(^{(cc)}X)^{\alpha} + A_{\alpha}^{\beta}(^{(cc)}X)^{\alpha}
\]
\[
= X^{\alpha}t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})A_{\alpha\beta}^{\beta} + X^{\alpha}t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})A_{\alpha\beta}^{\beta}
\]
\[
+ A_{\alpha}^{\beta}A_{\alpha\beta}^{\beta}t^{\alpha\beta}(\partial_{\alpha}X^{\alpha})+ A_{\alpha}^{\beta}A_{\alpha\beta}^{\beta}t^{\alpha\beta}(\partial_{\alpha}X^{\alpha})
\]
\[
= \sum_{q=1}^{4} b_{q}
\]
where,
\[
b_1 = X^{\alpha}t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})A_{\alpha\beta}^{\beta}
\]
\[
b_2 = X^{\alpha}t^{\alpha\beta}(\partial_{\alpha}A_{\beta\alpha}^{\beta})A_{\alpha\beta}^{\beta}
\]
\[
b_3 = A_{\alpha}^{\beta}A_{\alpha\beta}^{\beta}t^{\alpha\beta}(\partial_{\alpha}X^{\alpha})
\]
\[
b_4 = A_{\alpha}^{\beta}A_{\alpha\beta}^{\beta}t^{\alpha\beta}(\partial_{\alpha}X^{\alpha})
\]
You can check that
\[
a_1 = b_3, a_2 = b_1, a_3 = b_2, a_4 = b_4.
\]
Thus, we have (3.1).

4. Horizontal lifts of vector fields

Let \( X \in \mathcal{S}_{0}^{1}(T(M_{n})) \), i.e. \( X = X^{\alpha} \partial_{\alpha} \). If we take account of (1.2), we can prove that \( ^{HH}X' \in \mathcal{S}_{0}^{1}(T(M_{n})) \) is a vector field defined by
\[
^{HH}X = \begin{pmatrix}
-\Gamma_{\alpha}^{\beta}X^{\alpha} \\
X^{\beta} \\
-\Gamma_{\alpha}^{\beta}t^{\alpha\epsilon}X^{\sigma} - \Gamma_{\alpha}^{\beta}t^{\alpha\epsilon}X^{\sigma}
\end{pmatrix}
\]
with respect to the coordinates \((\vec{\alpha}, \chi^{\beta}, \vec{\beta})\) on \( T_{0}^{1}(M_{n}) \). We call \( ^{HH}X \) the horizontal lift of the vector field \( X \) to \( T_{0}^{1}(M_{n}) \). Where
\[
\Gamma_{\alpha}^{\beta} = \gamma^{\epsilon} \Gamma_{\alpha}^{\beta} 
\]

**Theorem 4.1.** If \( X \in \mathcal{S}_{0}^{1}(T(M_{n})) \) then
\[
^{cc}X - ^{HH}X = \chi(\tilde{\nabla}X) + \gamma(\nabla X),
\]
where the symmetric affine connection \( \tilde{\nabla} \) is given by \( \Gamma_{\alpha}^{\alpha} = \Gamma_{\alpha}^{\alpha} \).
Proof. From (2.2), (2.3), (3.1) and (4.1), we have

\[
\tag{5.1}
\begin{align*}
\ell X - H X = & \left( \frac{\chi^\beta}{X^\beta} \frac{\partial X^\alpha}{\partial x^\alpha} + \frac{\gamma^\alpha}{\gamma^\alpha} \frac{\partial X^\alpha}{\partial \gamma^\alpha} \right) - \left( \frac{-\Gamma^\beta_\alpha}{X^\beta} \frac{\partial X^\alpha}{\partial x^\alpha} + \frac{\Gamma^\alpha_\beta}{\gamma^\alpha} \frac{\partial X^\beta}{\partial \gamma^\alpha} \right) \\
& = \left( \frac{\gamma^\alpha}{\gamma^\alpha} \frac{\partial X^\alpha}{\partial x^\alpha} + \frac{\gamma^\beta}{\gamma^\beta} \frac{\partial X^\beta}{\partial \gamma^\beta} \right) \\
& = \left( \frac{\gamma^\alpha}{\gamma^\alpha} \frac{\partial X^\alpha}{\partial x^\alpha} + \frac{\gamma^\beta}{\gamma^\beta} \frac{\partial X^\beta}{\partial \gamma^\beta} \right) \\
& = \gamma \left( \tilde{\nabla} X + \nabla X \right)
\end{align*}
\]

which prove Theorem 4.1. \(\square\)

5. Cross-sections in the semi-tensor bundle of the type (2,0)

Let \(\xi \in \mathcal{G}_1(M_n)\) be a tensor field of the type (2,0) on \(M_n\). Then the correspondence \(x \to \xi_x\), \(\xi_x\) being the value of \(\xi\) at \(x \in T(M_n)\), determines a cross-section \(\tilde{\beta}_\xi\) of \(T_2(M_n)\).

Thus if \(\sigma_\xi : M_n \to T_2(M_n)\) is a cross-section of \((T_2(M_n), \mathcal{G}_1(M_n))\), such that \(\tilde{\pi} \circ \sigma_\xi = \iota(M_n)\), an associated cross-section \(\beta_\xi : T(M_n) \to T_2(M_n)\) of semi-tensor bundle \((T_2(M_n), \pi_2, T(M_n))\) defined by \([11], p. 217-218\), \([5], [6], [13], p. 122\):

\[
\beta_\xi \left( \xi, x^\alpha \right) = \left( \xi, x^\alpha, \sigma_\xi \circ \pi_1 \left( \xi, x^\alpha \right) \right) = \left( \xi, x^\alpha, \sigma_\xi \left( x^\alpha \right) \right) = \left( \xi, x^\alpha, \xi^\alpha_{\xi^\alpha} \left( x^\beta \right) \right).
\]

If the \((2,0)\) tensor field \(\xi\) has the local components \(\xi^\alpha_{\xi^\alpha} \left( x^\beta \right)\), the cross-section \(\beta_\xi \left( T(M_n) \right)\) of \(T_2(M_n)\) is locally expressed by

\[
\begin{align*}
\left( \xi, x^\alpha, \xi^\beta, \xi^\alpha_{\xi^\alpha} \left( x^\beta \right) \right) \quad \text{with respect to the coordinates} \quad x^\beta = \left( x^\alpha, x^\beta, x^\beta \right) \quad \text{in} \quad T_2(M_n).
\end{align*}
\]

\(\xi^\alpha = \gamma^\alpha\) being considered as parameters. Differentiating (5.1) by \(\gamma^\alpha = \gamma^\alpha\), we have vector fields \(B_\gamma (\overline{\gamma} = 1, \ldots, n)\) with components

\[
B_\gamma = \begin{pmatrix} \frac{\partial x^\gamma}{\partial x^\alpha} + \frac{\partial \gamma^\gamma}{\partial x^\alpha} \frac{\partial x^\gamma}{\partial \gamma^\alpha} \end{pmatrix},
\]

which are tangent to the cross-section \(\beta_\gamma \left( T(M_n) \right)\).

Thus \(B_\gamma\) have components

\[
B_\gamma \left( B_\gamma \right) = \begin{pmatrix} \delta^\gamma_\gamma & 0 \\ 0 & 0 \end{pmatrix}.
\]
with respect to the coordinates \((x^\alpha, x^\beta, \ldots, x^\theta)\) in \(r_0^2(M_n)\). Where
\[
\alpha^\beta = A^\beta_x = \frac{\partial x^\beta}{\partial x^\alpha}.
\]

Let \(X \in \mathcal{S}_0^1(T(M_n))\), i.e. \(X = X^\alpha \partial_\alpha\). We denote by \(BX\) the vector field with local components
\[
BX : \left( B^\alpha_x X^\alpha \right) = \begin{pmatrix}
\frac{\partial B^\alpha_x}{\partial x^\alpha} X^\alpha \\
0
\end{pmatrix} = \begin{pmatrix}
A^\beta_x X^\alpha \\
0
\end{pmatrix} = \begin{pmatrix}
X^\beta \\
0
\end{pmatrix}
\]

with respect to the coordinates \((x^\alpha, x^\beta, \ldots, x^\theta)\) in \(r_0^2(M_n)\), which is defined globally along \(\beta_x(T(M_n))\).

Differentiating (5.1) by \(x^\theta\), we have vector fields \(C_{(\theta)} (\theta = n + 1, \ldots, 2n)\) with components
\[
C_{(\theta)} = \frac{\partial x^\beta}{\partial x^\theta} = \partial_\theta x^\beta = \begin{pmatrix}
\frac{\partial x^\beta}{\partial x^\theta} \\
\frac{\partial x^\beta}{\partial x^\theta}
\end{pmatrix} ,
\]

which are tangent to the cross-section \(\beta_x(T(M_n))\). Thus \(C_{(\theta)}\) have components
\[
C_{(\theta)} : \left( C_{(\theta)}^\beta \right) = \begin{pmatrix}
\frac{\partial x^\beta}{\partial x^\theta} \\
\frac{\partial x^\beta}{\partial x^\theta}
\end{pmatrix} ,
\]

with respect to the coordinates \((x^\alpha, x^\beta, \ldots, x^\theta)\) in \(r_0^2(M_n)\). Where
\[
\delta^\alpha_\beta = \delta^\alpha_\beta = \frac{\partial x^\beta}{\partial x^\alpha}.
\]

Let \(X \in \mathcal{S}_0^1(T(M_n))\). Then we denote by \(CX\) the vector field with local components
\[
CX : \left( C_{(\theta)}^\beta X^\beta \right) = \begin{pmatrix}
X^\theta \partial_\theta x^\beta \\
X^\beta \\
X^\beta \partial_\theta x^\beta
\end{pmatrix} \begin{pmatrix}
\delta^\alpha_\beta \\
\delta^\alpha_\beta \\
\delta^\alpha_\beta
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

with respect to the coordinates \((x^\alpha, x^\beta, \ldots, x^\theta)\) in \(r_0^2(M_n)\), which is defined globally along \(\beta_x(T(M_n))\).

On the other hand, the fibre is locally expressed by
\[
\begin{align*}
x^\theta &= \text{const.} , \\
x^\beta &= \text{const.} , \\
x^\delta &= \tau^{\alpha_1 \alpha_2} = \tau^{\alpha_1 \alpha_2} ,
\end{align*}
\]

\(\tau^{\alpha_1 \alpha_2}\) being considered as parameters. Thus, on differentiating with respect to \(x^\beta = \tau^{\alpha_1 \alpha_2}\), we easily see that the vector fields \(E_{(\beta)}^{(\alpha)}\) \((\beta = 2n + 1, \ldots, 2n + n^2)\) with components
\[
E_{(\beta)}^{(\alpha)} : \left( E_{(\beta)}^{(\alpha)} \right) = \begin{pmatrix}
\partial_\beta x^\alpha \\
\partial_\beta x^\alpha \\
\partial_\beta x^\alpha
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

is tangent to the fibre, where \(\partial\) is the Kronecker symbol.

Let \(\xi\) be a tensor field of the type \((2, 0)\) with local components
\[
\xi = \xi^{\gamma \gamma} \partial_\gamma \otimes \partial_\gamma
\]
on \(M_n\).

We denote by \(E \xi\) the vector field with local components
\[
E \xi : \left( E \xi \right) = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} ,
\]

which is tangent to the fibre.

**Theorem 5.1.** Let \(X\) be a vector field on \(T(M_n)\), we have along \(\beta_x(T(M_n))\) the formula
\[
^\alpha X = CX + B(L_V X) + E(-L_X \xi) ,
\]

where \(L_V X\) denotes the Lie derivative of \(X\) with respect to \(V\), and \(L_X \xi\) denotes the Lie derivative of \(\xi\) with respect to \(X\).
Proof. Using (3.1), (5.2), (5.3) and (5.4), we have

\[
CX + B(L_x X) + E(-L_x \xi) = \begin{pmatrix}
X^\theta \partial_\theta V^\beta \\
X^\beta \\
X^\theta \partial_\theta \xi_{\alpha_1 \alpha_2}
\end{pmatrix} + \begin{pmatrix}
V^\alpha \partial_\alpha X^\beta - X^\alpha \partial_\alpha V^\beta \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
- X^\theta \partial_\eta \xi_{\alpha_1 \alpha_2} + \xi_{\alpha_1 \alpha_2} \partial_\eta X^\alpha + \xi_{\alpha_1 \alpha_2} \partial_\eta X^\alpha \\
0 \\
0
\end{pmatrix} = \epsilon^c X.
\]

Thus, we have Theorem 5.1.

On the other hand, on putting \( C(\bar{B}) = E(\bar{B}) \), we write the adapted frame of \( \beta_\xi(T(M_n)) \) as \( \{B(\bar{B}), C(\bar{B})\} \). The adapted frame \( \{B(\bar{B}), C(\bar{B})\} \) of \( \beta_\xi(T(M_n)) \) is given by the matrix

\[
\bar{A} = \begin{pmatrix}
\delta^\alpha_\beta \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
\delta^\alpha_\beta \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
\delta^\alpha_\beta \\
0 \\
0
\end{pmatrix} \text{ (5.5)}.
\]

Since the matrix \( \bar{A} \) in (5.5) is non-singular, it has the inverse. Denoting this inverse by \( \bar{A}^{-1} \), we have

\[
\bar{A}^{-1} = \begin{pmatrix}
\delta^\beta_\alpha & \partial_\beta V^\alpha & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \text{ (5.6)}.
\]

where \( \bar{A}^{-1} = (\bar{A}_\xi^{-1}) \), \( \bar{A}_\xi^{-1} = \bar{A}_\xi - \bar{I} \), where \( \bar{A} = (\bar{B}, \alpha, \bar{B}), B = (\bar{B}, \beta, \bar{B}), C = (\bar{B}, \theta, \bar{B}) \).

Proof. In fact, from (5.5) and (5.6), we easily see that

\[
\bar{A}^{-1} = \begin{pmatrix}
\delta^\beta_\alpha & \partial_\beta V^\alpha & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
\delta^\beta_\alpha & \partial_\beta V^\alpha & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} = \bar{A}_\xi = \bar{I}.
\]

Then we see from Theorem 5.1 that the complete lift \( \epsilon^c X \) of a vector field \( X \in \mathfrak{g}(T(M_n)) \) along \( \beta_\xi(T(M_n)) \) components of the form

\[
\epsilon^c X : \begin{pmatrix}
L_x X \\
X \\
- L_x \xi
\end{pmatrix},
\]

with respect to the adapted frame \( \{B(\bar{B}), C(\bar{B})\} \).

Let \( A \in \mathfrak{g}(T(M_n)) \). If we take account of (2.1) and (5.5), we can easily prove that \( v^\nu A = \bar{A}(v^\nu A) \), where \( v^\nu A \in \mathfrak{g}_0(T(M_n)) \) is a vector field defined by

\[
v^\nu A = \begin{pmatrix}
v^\nu A^\theta \\
v^\nu A^\alpha \\
v^\nu A^\beta
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} \text{ (5.5)}.
\]

with respect to the adapted frame \( \{B(\bar{B}), C(\bar{B})\} \) of \( \beta_\xi(T(M_n)) \).

Let \( \varphi \in \mathfrak{g}_0(T(M_n)) \) now. If we take account of (2.2) and (5.5), we see that \( (\gamma \varphi)^\nu = \bar{A}(\gamma \varphi) \). \( \varphi \) is given by

\[
\gamma \varphi = (\gamma \varphi)^\nu = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \text{ (5.5)}.
\]
with respect to the adapted frame \( \{ B(\beta), C(\beta), C(\beta^T) \} \).

\( BX, CX \) and \( E_\xi \) also have components:

\[
BX = \begin{pmatrix}
X^\alpha \\
0 \\
0
\end{pmatrix},
CX = \begin{pmatrix}
0 \\
X^\alpha \\
0
\end{pmatrix},
E_\xi = \begin{pmatrix}
0 \\
0 \\
\xi_{,\alpha_1\alpha_2}
\end{pmatrix}
\]

respectively, with respect to the adapted frame \( \{ B(\beta), C(\beta), C(\beta^T) \} \) of the cross-section \( \beta_\xi (T(M_n)) \) determined by a tensor field \( \xi \) of the type \((2,0)\) in \( T(M_n) \).

References