Coefficient Estimates for Certain Subclasses of m-fold Symmetric Bi-univalent Functions Defined by the Q-derivative Operator

F. Müge Sakar1* and H. Özlem Günü2

1Batman University, Faculty of Economics and Administrative Sciences, Batman-Turkey
2Dicle University, Faculty of Science, Department of Mathematics, Diyarbakır-Turkey
*Corresponding author E-mail: mugesakar@hotmail.com

Abstract

In the present study, we introduce two new subclasses of bi-univalent functions based on the q-derivative operator in which both \( f \) and \( f^{-1} \) are m-fold symmetric analytic functions in the open unit disk. Among other results belonging to these subclasses upper coefficients bounds \( |a_{m+1}| \) and \( |a_{2m+1}| \) are obtained in this study. Certain special cases are also indicated.

Keywords: Analytic functions, univalent functions, bi-univalent functions

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1. Introduction

Let \( \mathcal{A} \) denote the family of functions analytic in the open unit disk \( D = \{ z : z \in \mathbb{C}, |z| < 1 \} \) and normalized by the conditions \( f(0) = f'(0) - 1 = 0 \) and having the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k.
\]

A function is said to be univalent if it never takes the same value twice, that is \( f(z_1) = f(z_2) \) if \( z_1 \neq z_2 \). We also denote by \( \mathcal{S} \) the subclass of functions in \( \mathcal{A} \) which are univalent in \( D \) (see for details [7]). From the Koebe 1/4 Theorem (for details, see [7]) every univalent function \( f \) has an inverse \( f^{-1} \) satisfying

\[
f^{-1}(f(z)) = z \quad (z \in D)
\]

and

\[
f\left(f^{-1}(w)\right) = w \quad \left( |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4} \right).
\]

In fact, the inverse function \( f^{-1} \) is given by

\[
g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots
\]

\[
= w + \sum_{k=2}^{\infty} b_k w^k.
\]

Let \( f \in \mathcal{A} \). The function \( f \) is said to be bi-univalent in \( D \) if both \( f \) and \( f^{-1} \) are univalent in \( D \). Let \( \Sigma \) denote the class of bi-univalent functions in \( D \) given by the Taylor-Maclaurin series expansion given by (1.1). We can accept that the beginning of estimating bounds for the coefficients of classes of bi-univalent functions is the date 1967 [11]. Later the papers of Brannan and Taha [4] and Srivastava et al. [20] were picked up the interest on the coefficient bounds of bi-univalent functions.

Email addresses: mugesakar@hotmail.com (F. Müge Sakar), ozlemg@dicle.edu.tr (H. Özlem Günü)
For detailed information about the class of \( \Sigma \) was given in the references [4], [11], [14], [20] and [23]. Let \( m \in \mathbb{N} = \{1, 2, 3, \ldots \} \). A domain \( \mathbb{E} \) is said to be \( m \)-fold symmetric if a rotation of \( \mathbb{E} \) about the origin through an angle \( 2\pi/m \) carries \( \mathbb{E} \) on itself. It follows that, a function \( f \) analytic in \( \mathbb{D} \) is said to be \( m \)-fold symmetric if

\[
f(z) = f(e^{2\pi i/m}z) = e^{2\pi i/m}f(z).
\]

In particular every \( f \) is one-fold symmetric and every odd \( f \) is two-fold symmetric. \( \mathcal{S}_m \) indicate the class of \( m \)-fold symmetric univalent functions in \( \mathbb{D} \).

\( f \in \mathcal{S}_m \) is characterized by having a power series as following normalized form

\[
f(z) = z + \sum_{k=1}^{m} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{D}, \ m \in \mathbb{N}).
\]  

(1.3)

In [21] Srivastava et al. defined \( m \)-fold symmetric bi-univalent function analogues to the concept of \( m \)-fold symmetric univalent functions. They introduce some important results, such as each function \( f \in \Sigma \) generates an \( m \)-fold symmetric bi-univalent function for each \( (m \in \mathbb{N}) \). In addition, they acquired the series expansion for \( f^{-1} \) as follows:

\[
g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right]w^{2m+1}
\]

\[
= z + \sum_{k=1}^{m} A_{mk+1} z^{mk+1}
\]  

(1.4)

where \( f^{-1} = g \). We denote by \( \Sigma_m \) the class of \( m \)-fold symmetric bi-univalent functions in \( \mathbb{D} \). For some examples of \( m \)-fold symmetric bi-univalent functions, see [21]. The coefficient problem for \( m \)-fold symmetric analytic bi-univalent functions is one of the favorite subjects of geometric function theory in these days, see [1], [2], [5], [8], [21], [22]. Here, the aim of this study is to determine upper coefficients bounds \( |a_{m+1}| \) and \( |a_{2m+1}| \) are obtained belonging these two new subclasses.

First formulae in what we now call \( q \)-calculus were obtained by Euler in the eighteenth century. In the second half of the twentieth century there was a significant increase of activity in the area of the \( q \)-calculus. The fractional calculus operators has gained importance and popularity, mainly due to its vast potential of demonstrated applications in various fields of applied sciences, engineering. The application of \( q \)-calculus was initiated by Jackson [9].

In the field of geometric function theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional \( q \)-calculus is the important tools that are used to investigate subclasses of analytic functions. Historically speaking, a firm footing of the usage of the the \( q \)-calculus in the context of geometric function theory was actually provided and the basic (or \( q \)-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [19]). In fact, the extension of the theory of univalent functions can be described by using the theory of \( q \)-calculus. Furthermore, the \( q \)-calculus operators, such as fractional \( q \)-integral and fractional \( q \)-derivative operators, are used to construct several subclasses of analytic functions (see, [13], [15], [16]). In a recent paper Purohit and Raina [18] investigated applications of fractional \( q \)-calculus operators to defined certain new classes of functions which are analytic in the open disk. Later, Mohammed and Darus [12] studied approximation and geometric properties of these \( q \)-operators in some subclasses of analytic functions in compact disk. A comprehensive study on applications of \( q \)-calculus in operator theory may be found in [3]. For the convenience, we give some basic definitions and concept details of \( q \)-calculus which are used in this paper.

For a function \( f \in \mathcal{A} \) given by (1.1) and \( 0 < q < 1 \), the \( q \)-derivative of function \( f \) is defined by (see [7], [10])

\[
D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \neq 0)
\]  

(1.5)

\[
D_q f(0) = f'(0) \quad \text{and} \quad D_q^2 f(z) = D_q(D_q f(z)). \quad \text{From (1.5), we deduce that}
\]

\[
D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \quad \text{where}
\]

\[
[k]_q = \frac{1 - q^k}{1 - q}.
\]  

(1.7)

As \( q \to 1^- \), \( [k]_q \to k \), for a function \( g(z) = z^k \) we get

\[
D_q (z^k) = [k]_q z^{k-1},
\]
where \( g' \) is the ordinary derivative.

By making use of the \( q \)-derivative of a function \( f \in \mathcal{A} \), we introduce two new subclasses of the function class \( \Sigma_m \) and obtain estimates on the coefficients \( |a_{m+1}| \) and \( |a_{2m+1}| \) for functions in these new subclasses of the function class \( \Sigma_m \).

Firstly, in order to derive our main results, we need to following lemma.

**Lemma 1.1.** [17] If \( p \in \mathcal{P} \), then \( |c_k| \leq 2 \) for each \( k \) where \( \mathcal{P} \) is the family of all functions \( p \) analytic in \( \mathbb{D} \) for which

\[
\Re(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \cdots
\]

for \( z \in \mathbb{D} \).

### 2. Definition of the Class \( T_{\Sigma,m}^{q,\alpha} \) and Its Coefficient Bounds

**Definition 2.1.** A function \( f \) given by (1.3) is said to be in the class \( T_{\Sigma,m}^{q,\alpha} \) \((0 < q < 1, 0 < \alpha \leq 1, m \in \mathbb{N})\) if the following condition are satisfied

\[
f \in \Sigma_m \quad \text{and} \quad |\arg D_q f(z)| < \frac{\alpha \pi}{2} \quad (z \in \mathbb{D})
\]

(2.1) and

\[
|\arg D_q g(w)| < \frac{\alpha \pi}{2} \quad (w \in \mathbb{D})
\]

(2.2) where the function \( g \) is given by Eq.(1.4).

**Remark 2.2.** We note that \( \lim_{q \to 1} T_{\Sigma,m}^{q,\alpha} = T_{\Sigma,m}^\alpha \) and for one-fold case \( T_{\Sigma,1}^{q,\alpha} = T_{\Sigma,1}^\alpha \) introduced by Srivastava et al. [20].

**Theorem 2.3.** Let the function \( f \) given by (1.3) be in the function class \( T_{\Sigma,m}^{q,\alpha} \) \((0 < q < 1, 0 < \alpha \leq 1, m \in \mathbb{N})\). Then

\[
|a_{m+1}| \leq \frac{2\alpha}{\sqrt{(m+1)\alpha |1+2m| q - (\alpha - 1)(1+m)^2 q}}
\]

(2.3) and

\[
|a_{2m+1}| \leq \frac{2(m+1)\alpha^2}{1 + m^2 q} + \frac{2\alpha}{1 + 2m |q|}. \tag{2.4}
\]

Proof. First of all, it follows from the conditions (2.1) and (2.2) that

\[
D_q f(z) = |p(z)|^q, \quad \text{and} \quad D_q g(w) = |q(w)|^q. \quad (z, w \in \mathbb{D}) \tag{2.5}
\]

Respectively, where \( p(z) \) and \( q(z) \) are in familiar Caratheodory class \( \mathcal{P} \) (see for details [7]) and have the following series statement

\[
p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots
\]

(2.6) and

\[
q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots
\]

(2.7) Now, comparing the coefficients in (2.5), we have

\[
[1 + m |q|] a_{m+1} = \alpha p_m \tag{2.8}
\]

and

\[
[1 + 2m |q|] a_{2m+1} = \alpha p_{2m} + \frac{\alpha (\alpha - 1)}{2} p_m^2 \tag{2.9}
\]

\[
-[1 + m |q|] a_{m+1} = \alpha q_m \tag{2.10}
\]

\[
[1 + 2m |q|] a_{m+1}^2 - a_{2m+1} = \alpha q_{2m} + \frac{\alpha (\alpha - 1)}{2} q_m^2. \tag{2.11}
\]

From (2.8) and (2.9), we have
\[ p_m = -q_m \]  
(2.12) and

\[ 2[1 + m]^2 q_{m+1}^2 = \alpha^2(p_m^2 + q_m^2). \]  
(2.13)

Furthermore, from Eqs. (2.9), (2.11) and (2.13), we obtain that

\[ [1 + 2m]q(m + 1)a_{m+1}^2 = \alpha(p_{2m} + q_{2m}) + \frac{\alpha - 1}{\alpha} [1 + m]^2 q_{m+1}^2. \]

Therefore, we get

\[ a_{m+1}^2 = \frac{\alpha^2(p_{2m} + q_{2m})}{(m + 1)\alpha[1 + 2m]q - (\alpha - 1)[1 + m]^2 q}. \]  
(2.14)

Note that, according to the Caratheodory lemma [7], \(|p_m| \leq 2\) and \(|q_m| \leq 2\) for \(m \in \mathbb{N}\). Now taking the absolute value of (2.14) and applying the Caratheodory lemma for \(p_{2m}\) and \(q_{2m}\) we have the following inequality

\[ |a_{m+1}| \leq \frac{2\alpha}{\sqrt{(m + 1)\alpha[1 + 2m]q - (\alpha - 1)[1 + m]^2 q}}. \]

So, we obtain the desired estimate for \(|a_{m+1}|\) given by (2.3). Next, so as to obtain solution of the coefficient bound on \(|a_{2m+1}|\), we subtract (2.11) from (2.9). We thus have,

\[ 2[1 + 2m]q a_{2m+1} - (m + 1)[1 + 2m]q a_{m+1}^2 \]

\[ = \alpha(p_{2m} - q_{2m}) + \frac{\alpha(\alpha - 1)}{2}(p_m^2 - q_m^2). \]  
(2.15)

It follows from (2.13), (2.15) and observing \(p_m^2 - q_m^2\), it gives that

\[ a_{2m+1}^2 = \frac{\alpha(p_{2m} - q_{2m})}{2[1 + 2m]q} + \frac{(m + 1)\alpha^2(p_m^2 + q_m^2)}{4[1 + m]^2 q}. \]  
(2.16)

Taking the absolute value of (2.16) and applying Caratheodory lemma again for coefficients \(p_m, p_{2m}\) and \(q_{2m}\) we have

\[ |a_{2m+1}| \leq \frac{2(m + 1)\alpha^2}{[1 + m]^2 q} + \frac{2\alpha}{[1 + 2m]q}. \]

So the proof is completed.

**Remark 2.4.** For one-fold case, we note that \(T_{q,1}^{\alpha} H_{m}^p = H_{m}^{p,\alpha}\) introduced by Bulut [6].

Taking \(q \to 1^+\) in Theorem 2.1, we have the class, \(\lim_{q \to 1} T_{q,1}^{\alpha} H_{m}^p = H_{m}^{p,\alpha}\) introduced by Srivastava et al. [21] and obtain the Corollary 2.1 as follows:

**Corollary 2.5.** [21] Let the function \(f \in H_{m}^p, (0 < \alpha \leq 1, m \in \mathbb{N})\) be given (1.3). Then

\[ |a_{m+1}| \leq \frac{2\alpha}{\sqrt{(m + 1)(am + m + 1)}} \]

and

\[ |a_{2m+1}| \leq \frac{2\alpha^2}{m + 1} + \frac{2\alpha}{2m + 1}. \]

**Remark 2.6.** For one-fold case, we note that \(\lim_{q \to 1} T_{q,1}^{\alpha} H_{m}^p = H_{m}^{p,\alpha}\) and we can obtain the results of Srivastava et al.[20].
3. Definition of the Class $T_{\Sigma,m}^q(\beta)$ and Its Coefficient Bounds

**Definition 3.1.** A function $f$ given by (1.3) is said to be in the class $T_{\Sigma,m}^q(\beta)$, $(0 < q < 1, 0 \leq \beta < 1, m \in \mathbb{N})$ if the conditions given below are fulfilled:

\[ f \in \Sigma_m \text{ and } \Re \{D_q f(z)\} > \beta \quad (z \in \mathbb{D}) \]  
(3.1)

and

\[ \Re \{D_q g(w)\} > \beta \quad (w \in \mathbb{D}) \]  
(3.2)

where the function $g$ is given by Eq. (1.4).

**Remark 3.2.** Note that we have the class $\lim_{q \to 1} T_{\Sigma,m}^{q,\alpha} = T_{\Sigma,m}^{\alpha}$ and for one-fold case the class $\lim_{q \to 1} T_{\Sigma,1}^{q}(\beta) = T_{\Sigma}(\beta)$ introduced by Srivastava et al. [20].

**Theorem 3.3.** Let the function $f$ given by (1.3) be in the function class $T_{\Sigma,m}^q(\beta)$, $(0 < q < 1, 0 \leq \beta < 1, m \in \mathbb{N})$. Then

\[ |a_{m+1}| \leq \min \left\{ \frac{2(1 - \beta)}{1 + m}, \frac{2\sqrt{1 - \beta}}{(1 + 2m)p_q(m + 1)} \right\} \]  
(3.3)

and

\[ |a_{2m+1}| \leq \frac{2(1 - \beta)}{1 + 2mp_q}. \]  
(3.4)

**Proof.** First of all, it follows from the equations (3.1) and (3.2) that

\[ D_q f(z) = [p(z)]^q D_q g(w) = [q(w)]^q, \quad (z, w \in \mathbb{D}) \]  
(3.5)

respectively, where $p(z)$ and $q(z)$ given by (2.6) and (2.7). Now equating coefficients in (3.5), we obtain

\[ [1 + m]p_q a_{m+1} = (1 - \beta)p_m \]  
(3.6)

\[ [1 + 2m]p_q a_{2m+1} = (1 - \beta)p_{2m} \]  
(3.7)

\[ -(1 + m)q_m a_{m+1} = (1 - \beta)q_{m} \]  
(3.8)

\[ [1 + 2m]q_q (m + 1)a_{m+1}^2 - a_{2m+1} = (1 - \beta)q_{2m}. \]  
(3.9)

From Eqs. (3.6) and (3.8), we have

\[ p_m = -q_m \]  
(3.10)

and

\[ 2[1 + m]p_q a_{m+1}^2 = (1 - \beta)^2(p_m + q_m^2). \]  
(3.11)

Also, from Eqs. (3.7) and (3.9), we obtain

\[ [1 + 2m]q_q (m + 1)a_{m+1}^2 = (1 - \beta)(p_{2m} + q_{2m}). \]  
(3.12)

Thus, applying Caratheodory lemma for (3.11) and (3.12) we obtain the coefficient estimate $|a_{m+1}|$ as follows:

\[ |a_{m+1}^2| \leq \frac{1 - \beta}{1 + 2mp_q(m + 1)} \left( |p_{2m}| + |q_{2m}| \right) \]  
(3.13)

which is desired coefficient bound. Next, so as to obtain bound for coefficient $|a_{2m+1}|$ by subtracting (3.9) from (3.7), we have

\[ -(1 + 2m)q_q (m + 1)a_{m+1}^2 + 2[1 + 2m]q_q a_{2m+1} = (1 - \beta)(p_{2m} - q_{2m}) \]  
(3.14)

or equivalently

\[ a_{2m+1} = \frac{(1 - \beta)(p_{2m} - q_{2m})}{2[1 + 2mp_q]} + \frac{m + 1}{2} a_{m+1}^2. \]
Upon substituting the value of $a_{m+1}^2$ from (3.11), we obtain
\begin{equation}
\begin{aligned}
a_{2m+1} = \frac{(1 - \beta)(p_{2m} - q_{2m})}{2[1 + 2m]_q} + \frac{(m + 1)(1 - \beta)^2(p_m^2 + q_m^2)}{4[1 + m]_q^2}.
\end{aligned}
\tag{3.14}
\end{equation}

Applying Caratheodory lemma for coefficients $p_m, q_m, p_{2m}$ and $q_{2m}$ we obtain
\begin{equation}
\begin{aligned}
|a_{2m+1}| &\leq \frac{2(1 - \beta)}{1 + 2m} + \frac{2(m + 1)(1 - \beta)^2}{1 + m}.
\end{aligned}
\end{equation}

On the other hand, by using the equation (3.12) into (3.13), and applying Caratheodory lemma we can obtain the inequality as follows
\begin{equation}
\begin{aligned}
|a_{2m+1}| &\leq \frac{2(1 - \beta)}{1 + 2m}.
\end{aligned}
\end{equation}

which is the desired bounds on coefficients $|a_{2m+1}|$ as given in Theorem 3.1.

Taking $q \to 1$ in Theorem 3.3, we obtain following corollary.

**Corollary 3.4.** Let the function $f$ given by (1.3) be in the class $T_{E,m}(\beta)$, $(0 \leq \beta < 1, m \in \mathbb{N})$, Then
\begin{equation}
\begin{aligned}
|a_{m+1}| &\leq \left\{ \frac{2\sqrt{1 - \beta}}{(m + 1)(m + 2)} : 0 \leq \frac{m}{1 + 2m}, \frac{2(1 - \beta)}{1 + m} \leq \beta < 1 \right\}
\end{aligned}
\end{equation}

and
\begin{equation}
\begin{aligned}
|a_{2m+1}| &\leq \frac{2(1 - \beta)}{1 + 2m}.
\end{aligned}
\end{equation}

**Remark 3.5.** For one fold case, Corollary 3.1 reduces to the following corollary given by Bulut [6] for the bounds on coefficients $|a_2|$ and $|a_3|$.

**Corollary 3.6.** [6] Let the function $f$ given by Taylor-Maclaurin series expansion (1.1) be in the class $H_{E}(\beta)$, $(0 \leq \beta < 1)$. Then
\begin{equation}
\begin{aligned}
|a_2| &\leq \left\{ \sqrt{\frac{2(1 - \beta)}{3}} : 0 \leq \beta < \frac{1}{3} \right\}
\end{aligned}
\end{equation}

and
\begin{equation}
\begin{aligned}
|a_3| &\leq \frac{2(1 - \beta)}{3}.
\end{aligned}
\end{equation}

**Remark 3.7.** Corollary 3.2 given above is an improvement of the estimates for coefficients on $|a_2|$ and $|a_3|$ obtained by Srivastava et al [20].

**Corollary 3.8.** [20] Let the function $f$ given by Taylor-Maclaurin series expansion (1.1) be in the class $H_{E}(\beta)$, $(0 \leq \beta < 1)$. Then
\begin{equation}
\begin{aligned}
|a_2| &\leq \sqrt{\frac{2(1 - \beta)}{3}} \quad \text{and} \quad |a_3| \leq \frac{(1 - \beta)(5 - 3\beta)}{3}.
\end{aligned}
\end{equation}

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**References**


