Coefficient estimates for a subclass of analytic bi-pseudo-starlike functions of Ma-Minda type

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Abstract

In this paper, we introduce a new subclass \( \mathcal{L}_{\lambda}^{\Sigma}(\phi) \) of analytic and bi-univalent functions in the open unit disk \( U \). For functions belonging to this class, we obtain initial coefficient bounds. Our results generalize and improve some earlier results in the literature.

Keywords: Analytic functions; univalent functions; bi-univalent functions; coefficient bounds; subordination; pseudo-starlike functions.

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1. Introduction

Let \( R = (-\infty, \infty) \) be the set of real numbers, \( \mathbb{C} \) be the set of complex numbers and
\[ \mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\} \]
be the set of positive integers.

Let \( \mathcal{A} \) denote the class of all functions of the form
\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \]
which are analytic in the open unit disk
\[ U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\} \).

We also denote by \( \mathcal{S} \) the class of all functions in the normalized analytic function class \( \mathcal{A} \) which are univalent in \( U \).

For two functions \( f \) and \( g \), analytic in \( U \), we say that the function \( f \) is subordinate to \( g \) in \( U \), and write
\[ f(z) \prec g(z) \quad (z \in U), \]
if there exists a Schwarz function \( \omega \), which is analytic in \( U \) with
\[ \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U) \]
such that
\[ f(z) = g(\omega(z)) \quad (z \in U). \]

Indeed, it is known that
\[ f(z) \prec g(z) \quad (z \in U) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence
\[ f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]
We observe that $B$ where all coefficients are real and $f$.

We suppose also that $\beta \in \mathbb{R}$ in the unit disk $U$.

In fact, the inverse function $f^{-1}$ is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_1^2 - a_3\right) w^3 - \left(5a_1^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots.$$  \hspace{1cm} (1.2)

A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ denote the class of bi-univalent functions in $U$.

Recently, Babalola [3] defined the class $\mathcal{L}_\lambda(\beta)$ of $\lambda$-pseudo-starlike functions of order $\beta$ as follows:

Suppose $0 \leq \beta < 1$ and $\lambda \geq 1$ is real. A function $f \in A$ given by (1.1) belongs to the class $\mathcal{L}_\lambda(\beta)$ of $\lambda$-pseudo-starlike functions of order $\beta$ in the unit disk $U$ if and only if

$$\Re \left( \frac{z(f'(z))^\lambda}{f(z)} \right) > \beta \quad (z \in U).$$

Babalola [3] proved that all pseudo-starlike functions are Bazilević of type $1 - 1/\lambda$, order $\beta^{1/\lambda}$ and univalent in $U$.

Motivated by the abovementioned works, we define the following subclass of function class $\Sigma$.

**Definition 1.1.** For $\lambda \geq 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{L}_\lambda^1(\varphi)$ if the following conditions are satisfied:

$$\frac{z(f'(z))^\lambda}{f(z)} < \varphi(z) \quad (z \in U)$$

and

$$w q'(w) \frac{\lambda}{g(w)} < \varphi(w) \quad (w \in U),$$

where the function $g = f^{-1}$ is defined by (1.2).
Remark 1.2. In the following special cases of Definition 1.1, we show how the class of analytic bi-univalent functions \( \mathcal{L} \mathcal{B}_{2}^{k}(\varphi) \) for suitable choices of \( \lambda \) and \( \varphi \) lead to certain known classes of analytic bi-univalent functions studied earlier in the literature.

(i) For \( \lambda = 1 \), we get the class \( \mathcal{L} \mathcal{B}_{2}^{1}(\varphi) = \mathcal{L} \mathcal{B}_{2}^{1}(\varphi) \) of Ma-Minda bi-starlike functions introduced and studied by Ali et al. \cite{1}.

(ii) If we let
\[
\varphi(z) := \varphi_{\alpha}(z) = \left( \frac{1 + z}{1 - z} \right)^{\alpha} = 1 + 2\alpha z + 2\alpha^{2}z^{2} + \cdots \quad (0 < \alpha \leq 1, z \in \mathbb{U}),
\]
then the class \( \mathcal{L} \mathcal{B}_{2}^{1}(\varphi) \) reduces to the class denoted by \( \mathcal{L} \mathcal{B}_{2}^{1}(\alpha) \) which is the subclass of the functions \( f \in \Sigma \) satisfying
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad \text{and} \quad \left| \arg \left( \frac{w(g'(w))^{3}}{g(w)} \right) \right| < \frac{\alpha\pi}{2},
\]
where the function \( g = f^{-1} \) is defined by \( \varphi(z) \).

(iii) If we let
\[
\varphi(z) := \varphi_{\beta}(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^{2} + \cdots \quad (0 \leq \beta < 1, z \in \mathbb{U}),
\]
then the class \( \mathcal{L} \mathcal{B}_{2}^{1}(\varphi) \) reduces to the class denoted by \( \mathcal{L} \mathcal{B}_{2}^{1}(\lambda, \beta) \) which is the subclass of the functions \( f \in \Sigma \) satisfying
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta \quad \text{and} \quad \Re \left( \frac{w(g'(w))^{3}}{g(w)} \right) > \beta,
\]
where the function \( g = f^{-1} \) is defined by \( \varphi(z) \).

The classes \( \mathcal{L} \mathcal{B}_{2}^{1}(\alpha) \) and \( \mathcal{L} \mathcal{B}_{2}^{1}(\lambda, \beta) \) are introduced and studied by Joshi et al. \cite{10}. In the special case \( \lambda = 1 \), we get the classes \( \mathcal{L} \mathcal{B}_{2}^{1}(\alpha) = \mathcal{L} \mathcal{B}_{2}^{1}([\alpha]) \) and \( \mathcal{L} \mathcal{B}_{2}^{1}(1, \beta) = \mathcal{L} \mathcal{B}_{2}^{1}([\beta]) \) introduced and studied by Brannan and Taha \cite{2}.

In order to derive our main results, we need the following lemma.

Lemma 1.3. \cite{16} Let \( k, l \in \mathbb{R} \) and \( z_{1}, z_{2} \in \mathbb{C} \). If \( |z_{1}| < R \) and \( |z_{2}| < R \), then
\[
|(k + 1)z_{1} + (k - l)z_{2}| \leq \begin{cases} 2R|k| & , \quad |k| \geq |l| \\ 2R|l| & , \quad |k| \leq |l| \end{cases}
\]

2. Main Results

Theorem 2.1. Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion \( (1.1) \) be in the function class \( \mathcal{L} \mathcal{B}_{2}^{1}(\varphi) \) and \( \lambda \geq 1 \). Then
\[
|a_{2}| \leq \sqrt{\frac{B_{1}^{2}}{(2\lambda - 1)^{2}[2(2\lambda - 1)B_{1} + \lambda B_{1}^{2} - (2\lambda - 1)B_{2}]}} \quad \text{(2.1)}
\]
and
\[
|a_{3}| \leq \begin{cases} \sqrt{\frac{B_{1}}{(2\lambda - 1)^{2}}} & , \quad B_{1} \leq \frac{(2\lambda - 1)^{2}}{2\lambda - 1} \\ \sqrt{\frac{B_{1}}{(2\lambda - 1)^{2}}(2(2\lambda - 1)B_{1} + \lambda B_{1}^{2} - (2\lambda - 1)B_{2})} + \sqrt{\frac{B_{1}}{3\lambda - 1}} & , \quad B_{1} \geq \frac{(2\lambda - 1)^{2}}{2\lambda - 1} \end{cases} \quad \text{(2.2)}
\]

Proof. Let \( f \in \mathcal{L} \mathcal{B}_{2}^{1}(\varphi) \) and \( g = f^{-1} \) be defined by \( (1.2) \). Then there are analytic functions \( u, v : \mathbb{U} \to \mathbb{U} \), with \( u(0) = v(0) = 0 \), such that
\[
\frac{zf'(z)}{f(z)} = \varphi(u(z)) \quad \text{(2.3)}
\]
and
\[
\frac{w(g'(w))^{3}}{g(w)} = \varphi(v(w)) \quad \text{(2.4)}
\]

It follows from (1.7), (1.8), (2.3) and (2.4) that
\[
(2\lambda - 1)a_{2} = B_{1}p_{1} \quad \text{(2.5)}
\]
\[
(2\lambda - 4\lambda + 1)a_{3}^{2} + (3\lambda - 1)a_{3} = B_{1}p_{2} + B_{2}p_{1}^{2} \quad \text{(2.6)}
\]
\[-(2\lambda - 1)a_{2} = B_{1}q_{1} \quad \text{(2.7)}
\]
\[
(2\lambda^{2} + 2\lambda - 1)a_{3}^{2} - (3\lambda - 1)a_{3} = B_{1}q_{2} + B_{2}q_{1}^{2} \quad \text{(2.8)}
\]
From (2.5) and (2.7), we find that
\[ p_1 = -q_1 \] (2.9)
and
\[ 2(2\lambda - 1)^2 a_2^2 = B_1^2 \left( p_2^2 + q_2^2 \right). \] (2.10)
Also from (2.6), (2.8) and (2.10), we have
\[ a_2^2 = \frac{B_1^2 (p_2 + q_2)}{2(2\lambda - 1) |\lambda B_1^2 - (2\lambda - 1) B_2|}. \] (2.11)
In view of (2.9) and (2.11), together with (1.6), we get
\[ |a_2|^2 \leq \frac{B_1^2 \left(1 - |p_1|^2\right)}{(2\lambda - 1) |\lambda B_1^2 - (2\lambda - 1) B_2|}. \] (2.12)
Substituting (2.5) in (2.12) we obtain
\[ |a_2| \leq \sqrt{\frac{B_1^2}{(2\lambda - 1) \left( |\lambda B_1^2 - (2\lambda - 1) B_2| \right)}}. \] (2.13)
which is desired inequality (2.1).
On the other hand, by subtracting (2.8) from (2.6) and a computation using (2.9) finally lead to
\[ a_3 = a_2^2 + \frac{B_1 (p_2 - q_2)}{2(2\lambda - 1)}. \] (2.14)
From (1.6), (2.5), (2.9) and (2.14), it follows that
\[
|a_3| \leq |a_2|^2 + \frac{B_1}{2(3\lambda - 1)} \left( |p_2| + |q_2| \right)
\leq |a_2|^2 + \frac{B_1}{3\lambda - 1} \left( 1 - |p_1|^2 \right)
= \left( 1 - \frac{(2\lambda - 1)^2}{3\lambda - 1} \right) |a_2|^2 + \frac{B_1}{3\lambda - 1}. \] (2.15)
Substituting (2.5) and (2.13) in (2.15) we obtain the desired inequality (2.2). \qed

**Remark 2.2.** Theorem 2.1 is an improvement of the estimates obtained by Mazi and Altunkaya [11, Corollary 5].
If we take \( \lambda = 1 \) in Theorem 2.1, then we have the following Corollary 1.

**Corollary 1.** Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion (1.1) be in the function class \( \mathcal{F}_2(\psi) \). Then
\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{B_1 + B_1^2 - B_2}}
\]
and
\[
|a_3| \leq \begin{cases} 
B_1^2, & B_1 \leq \frac{1}{2} \\
\left( 1 - \frac{1}{2\alpha} \right) \frac{B_1^2}{B_1 + |B_1^2 - B_2|} + \frac{B_1}{2}, & B_1 \geq \frac{1}{2}.
\end{cases}
\]

**Remark 2.3.** Corollary 1 is an improvement of the estimates obtained by Mazi and Altunkaya [11, Corollary 4].
If we consider the function \( \varphi_{\alpha} \), defined in Remark 1.2 (ii), in Theorem 2.1, then we get the following consequence.

**Corollary 2.** Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion (1.1) be in the function class \( \mathcal{F} \mathcal{S}_2(\alpha) \) and \( \lambda \geq 1 \). Then
\[
|a_2| \leq \frac{2\alpha}{\sqrt{(2\lambda - 1)(2\lambda - 1 + \alpha)}}
\]
and
\[
|a_3| \leq \begin{cases} 
\frac{4\alpha^2}{(2\lambda - 1)^2}, & 0 < \alpha \leq \frac{(2\lambda - 1)^2}{2(3\lambda - 1)} \\
\left( 1 - \frac{(2\lambda - 1)^2}{2(3\lambda - 1)} \right) \frac{4\alpha^2}{(3\lambda - 1)(2\lambda - 1 + \alpha)} + \frac{3\alpha}{3\lambda - 1}, & \frac{(2\lambda - 1)^2}{2(3\lambda - 1)} \leq \alpha \leq 1.
\end{cases}
\]
Remark 2.4. Note that the coefficient estimates on $|a_3|$ in Corollary 2 is an improvement of the estimate obtained by Joshi et al. [10, Theorem 1].

If we take $\lambda = 1$ in Corollary 2, then we get the following consequence.

Corollary 3. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $S_\lambda^*\alpha$. Then

| $a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}}$

and

| $a_3| \leq \begin{cases} 
4\alpha^2, & 0 < \alpha \leq \frac{1}{4} \\
5\alpha^2 - \frac{1}{4\alpha}, & \frac{1}{4} \leq \alpha < 1 
\end{cases}$

If we consider the function $q_\beta$, defined in Remark 1.2(iii), in Theorem 2.1, then we get the following consequence.

Corollary 4. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $S_\lambda(\lambda, \beta)$ and $\lambda \geq 1$. Then

| $a_2| \leq \frac{2(1-\beta)}{\sqrt{(2\lambda - 1)(2\lambda - 1 + 2\lambda \beta - 1)}}$

and

| $a_3| \leq \begin{cases} 
\frac{4(1-\beta)^2}{2(\lambda - 1)(1-\beta)} + \frac{2(1-\beta)}{2\lambda - 1}, & 0 \leq \beta \leq 1 - \frac{(2\lambda - 1)^2}{2(3\lambda - 1)} \\
\frac{4(1-\beta)^2}{2(\lambda - 1)}, & 1 - \frac{(2\lambda - 1)^2}{2(3\lambda - 1)} \leq \beta < 1 
\end{cases}$

Remark 2.5. Note that Corollary 4 is an improvement of the estimates obtained by Joshi et al. [10, Theorem 2].

If we take $\lambda = 1$ in Corollary 4, then we get the following consequence.

Corollary 5. Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $S_\lambda^*(\beta)$. Then

| $a_2| \leq \begin{cases} 
\sqrt{2}(1-\beta), & 0 \leq \beta \leq \frac{1}{2} \\
\frac{2(1-\beta)}{\sqrt{2}\beta}, & \frac{1}{2} \leq \beta < 1 
\end{cases}$

and

| $a_3| \leq \begin{cases} 
\frac{5\beta}{2}, & 0 \leq \beta < \frac{1}{2} \\
\frac{(1-\beta)(3-2\beta)}{2\beta}, & \frac{1}{2} \leq \beta \leq \frac{1}{4} \\
4(1-\beta)^2, & \frac{1}{4} \leq \beta < 1 
\end{cases}$

References


