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Short Article

Kinematic Mapping in Semi-Euclidean 4-Space

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ABSTRACT

We study the some algebraic properties of matrix associated to Hamilton operators which is defined for semi-quaternions. The kinematic mapping corresponding to these operators in semi-Euclidean 4-space is the same as the kinematic mapping of Blaschke and Grünwald.

Keywords: *Hamilton operators, Quasi-elliptic geometry, Semi-quaternion*

Dört Buyutlu Semi-Oklidean Uzayda kinematik dönüşümler

ÖZET

Semi-kuaterniyonların Hamilton opratorlarına karşılık gelen matrislerin bazı cebirsel özeliklerin araştırdık. Bu opratorlara karşılık gelen dönüşümler kinematığı dört boyutlu semi oklıd uzayında, Blaşke ve Grünwald dönüşümler kinematığı aynıdır..

Anahtar Kelimeler: *Hamilton operatöleri, Kuasi eliptik geometri, Semi kuaterniyon*

I. INTRODUCTION

QUATERNIONS was first introduced by William R. Hamilton as a successor to complex numbers. The quaternions have been used in various areas of mathematics. A brief introduction of the semi-quaternions is provided in [5]. Dyachkova [1] has showed that the set of all invertible elements of semi-quaternions with the quaternion product is a Lie group. Also, she considered the degenerate scalar product $\langle q , p \rangle = a_0 b_0 + a_1 b_1$. Accordingly, the semi-quaternions algebra with this product has the 4-dimensional semi-Euclidean space structure with rank 2 semi-metric. In [2], the algebraic properties of semi-quaternions are studied and De-Moivre's and Euler's formulas for these quaternions are given.

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De Moivre's formula implies that there are uncountably many unit semi-quaternions satisfying for $q^n = 1$ for $n \geq 2$. The matrix associated with a semi-quaternion is studied and by De-Moivre's formula the $n-th$ power of such a matrix can be obtained [3]. In this paper, after a review of some fundamental properties of the semi- quaternions, algebraic properties of Hamilton operators of these quaternion are studied. By these operators, we get the kinematic mapping of Blaschke and Grünwald. The corresponding geometry is quasi-elliptic geometry.

II. PRELIMINARIES

Definition1. The group of motion of the Euclidean plane is denoted by OA_2. If we choose a Cartesian coordinate system, then a motion $\alpha \in OA_2$ has the form

$$x \mapsto \mathrm{M}.x + b, \text{ with } \mathrm{M} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \qquad (1)$$

In homogeneous coordinate, Equ. (1) reads

$$x\mathbb{R} \mapsto (A.x)\mathbb{R}, \qquad A = \begin{bmatrix} 1 & 0^T \\ b & \mathrm{M} \end{bmatrix}.$$

The kinematic mapping of Blashke and Grüwald is a correspondence between points of real projective three-space $\mathbf{P}^3$ and planar Euclidean motions. It is defined as:

$$\mathrm{d}\mathbb{R} \in \mathrm{P}^3 \mapsto \begin{bmatrix} d_0^2 + d_3^2 & 0 & 0 \\ 2(d_0 d_1 - d_2 d_3) & d_3^2 - d_0^2 & 2d_0 d_3 \\ 2(d_1 d_3 + d_0 d_2) & -2d_0 d_3 & d_3^2 - d_0^2 \end{bmatrix} \in OA_2.$$

Note that the image that image of a point with coordinate $(0, d_1, d_2, 0)$ is not a Euclidean motion. We therefore call the line $x_0 = x_3 = 0$ the absolute line and consider the kinematic mapping defined in projective space without the absolute line.

It is an elementary exercise to verify that a rotation with angle ϕ and center x_m, y_m corresponds to the point $(1, x_m, y_m, -\cot\frac{\phi}{2})\mathbb{R}$ and that the translation $x \mapsto x + b$ corresponds to the point $(0, b_2, -b_1, 2)$. This is illustrated in Fig. 1, which shows an affine part of projective three-space[4].

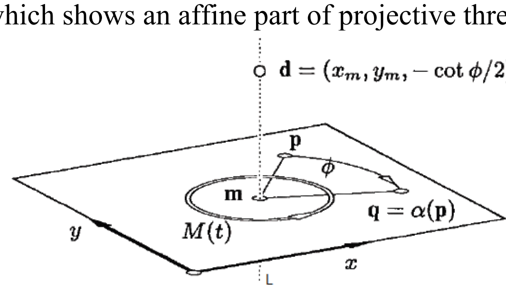

Fig. 1. A planar rotation α with center $\mathrm{m} = (x_m, y_m)$ transforms **p** to **q** has the kinematic image point **d**.

II. EXPERIMENT

A. SEMI-QUATERNIONS

This section summarizes the essentials of the algebra of semi-quaternions. A semi-quaternion q is an expression of the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^{\,2} = -1, \quad \vec{j}^{\,2} = \vec{k}^{\,2} = 0,$$
$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \quad \vec{j}\vec{k} = 0 = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \vec{j} = -\vec{i}\vec{k}.$$

The set of all semi-quaternions is denoted by H_s. We express the basic operations in terms of $\vec{i}, \vec{j}, \vec{k}$. The addition becomes as

$$(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) + (b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$
$$= (a_0 + b_0) + (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}$$

and the multiplication as

$$\begin{aligned}(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k})(b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k})\\ = (a_0b_0 - a_1b_1)\\ + (a_1b_0 + a_0b_1)\vec{i}\\ + (a_2b_0 + a_3b_1 + a_0b_2 - a_1b_3)\vec{j}\\ + (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)\vec{k}.\end{aligned}$$

Given $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, a_0 is called the *scalar part* of q, denoted by

$$S(q) = a_0,$$

and $a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ is called the *vector part* of q, denoted by

$$\vec{V}(q) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

If $S(q) = 0,$ then q is called pure semi-quaternion. The set of all the pure semi-quaternions is denoted by K.

The *conjugate* of q is

$$\overline{q} = a_0 - a_1\vec{i} - a_2\vec{j} - a_3\vec{k}.$$

The *norm* of q is

$$N_q = \overline{q}q = q\overline{q} = a_0^2 + a_1^2.$$

If $N_q = 1,$ then q is called a unit semi-quaternion.

The *inverse* of q with $N_q \neq 0,$ is

$$q^{-1} = \frac{1}{N_q}\bar{q}.$$

Clearly $qq^{-1} = 1 + 0\vec{i} + 0\vec{j} + 0\vec{k}.$ Note also that $\overline{qp} = \bar{p}\bar{q}$ and $(qp)^{-1} = p^{-1}q^{-1}.$ The algebra H_s has the 4-dimensional semi-Euclidean space structure $\mathbb{R}_2^4$ with rank 2 semi-metric[2].

B. MATRICES ASSOCIATED WITH SEMI-QUATERNIONS

We introduce the R-linear transformations representing left and right multiplication in H_s. Let q be a semi-quaternion. Then $L_q : \mathrm{H}_s \to \mathrm{H}_s$ and $R_q : \mathrm{H}_s \to \mathrm{H}_s$ are defined as follows:

$$L_q(x) = qx, \quad R_q = xq, \quad x \in \mathrm{H}_s.$$

If $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ then;

$$\begin{aligned}
L_q(1) &= a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, & R_q(1) &= a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \\
L_q(\vec{i}) &= -a_1 + a_0\vec{i} + a_3\vec{j} - a_2\vec{k}, & R_q(\vec{i}) &= -a_1 + a_0\vec{i} - a_3\vec{j} + a_2\vec{k} \\
L_q(\vec{j}) &= 0 + 0\vec{i} + a_0\vec{j} + a_1\vec{k}, & R_q(\vec{j}) &= 0 + 0\vec{i} + a_0\vec{j} - a_1\vec{k} \\
L_q(\vec{k}) &= 0 + 0\vec{i} - a_1\vec{j} + a_0\vec{k}, & R_q(\vec{k}) &= 0 + 0\vec{i} + a_1\vec{j} + a_0\vec{k}
\end{aligned}$$

Therefore the matrix representations of the linear operators L_q, R_q are, respectively

$$\Phi(q) = \begin{bmatrix} a_0 & -a_1 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \qquad (2)$$

and

$$\Psi(q) = \begin{bmatrix} a_0 & -a_1 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}. \qquad (3)$$

The Euler's and De-Moivre's formulae for the matrix A are studied in [3]. It is shown that as the De Moivre's formula implies, there are uncountably many matrices of unit quaternion satisfying $A^n = I_4$ for $n > 2.$

Theorem 1. If q and p are two semi-quaternions, λ is a real number and L_q and R_q are operators as defined in equations (2) and (3), respectively, then the following identities hold:

i. $q = p \Leftrightarrow \Phi(q) = \Phi(p) \Leftrightarrow \Psi(q) = \Psi(p).$

ii. $\Phi(q+p)=\Phi(q)+\Phi(p), \quad \Psi(q+p)=\Psi(q)+\Psi(p).$

iii. $\Phi(\lambda q)=\lambda\Phi(q), \ \Psi(\lambda q)=\lambda\Psi(q).$

iv. $\Phi(qp)=\Phi(q)\Phi(p), \quad \Psi(qp)=\Psi(p)\Psi(q).$

v. $\Phi(q^{-1})=\left[\Phi(q)\right]^{-1}, \quad \Psi(q^{-1})=\left[\Psi(q)\right]^{-1}, \quad N_q\neq 0.$

vi. $\det\left[\Phi(q)\right]=(N_q)^2, \quad \det\left[\Psi(q)\right]=(N_q)^2.$

vii. $tr\left[\Phi(q)\right]=4a_0, \quad tr\left[\Psi(q)\right]=4a_0.$

Proof: Identities (i), (ii) and (iii) can be proved easily. Using the associative property of the quaternions multiplication, it is clear that following identities hold:

$$(qp)r=q(pr)=qpr$$

In terms of operator Φ, the above identities can be written as

$$\begin{aligned}\Phi(qp)r &= \Phi(\Phi(q)p)r\\ &= \Phi(q)(\Phi(p)r)=\Phi(q)\Phi(p)r\end{aligned}$$

and similarly,

$$\begin{aligned}\Psi(qp)r &= \Psi(\Psi(q)p)r\\ &= \Psi(p)(\Psi(q)r)=\Psi(p)\Psi(q)r.\end{aligned}$$

Since r is arbitrary, the above relation employs equation (iv). Using the inverse property, we have

$$qq^{-1}=q^{-1}q=I_4$$

and in terms of operator Φ, the above identities can be written as

$$\Phi(qq^{-1})=\Phi(q)\Phi(q^{-1})=\Phi(I_4)=I_4,$$
$$\Psi(q^{-1}q)=\Psi(q^{-1})\Psi(q)=\Psi(I_4)=I_4,$$

therefore, the above relation employs equation (v). Identities (vi), and (vii) can be proved easily.

Theorem 2. Let q be a unit semi-quaternion. Matrices generated by operators $\Phi(q)$ and $\Psi(q)$ are semi-orthogonal matrices, *i.e.*

i) $\left[\Phi(q)\right]^T\varepsilon\Phi(q)=\varepsilon,$

ii) $\left[\Psi(q)\right]^T\varepsilon\Psi(q)=\varepsilon, \varepsilon=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$

Theorem 3. The map

$$\psi:(\mathrm{H}_s,+,.)\rightarrow(\mathrm{M}_{(4,\mathrm{R})},\oplus,\otimes)$$

defined as

$$\psi(a_0 + a_1\vec{\mathrm{i}} + a_2\vec{\mathrm{j}} + a_3\vec{\mathrm{k}}) \mapsto \begin{bmatrix} a_0 & -a_1 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix},$$

is an isomorphism of algebras.

Proof: We first demonstrate its homomorphic properties. If $p = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $q = b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ are any two semi-quaternions then:

$$\begin{aligned} \psi\{p+q\} &= \psi\left\{a_0 + b_0 + (a_1+b_1)\vec{i} + (a_2+b_2)\vec{j} + (a_3+b_3)\vec{k}\right\} \\ &= \begin{bmatrix} a_0+b_0 & -(a_1+b_1) & 0 & 0 \\ a_1+b_1 & a_0+b_0 & 0 & 0 \\ a_2+b_2 & a_3+b_3 & a_0+b_0 & -(a_1+b_1) \\ a_3+b_3 & -(a_2+b_2) & (a_1+b_1) & a_0+b_0 \end{bmatrix} \\ &= \begin{bmatrix} a_0 & -a_1 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \oplus \begin{bmatrix} b_0 & -b_1 & 0 & 0 \\ b_1 & b_0 & 0 & 0 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{bmatrix} \\ &= \psi\{p\} \oplus \psi\{q\}, \end{aligned}$$

$$\begin{aligned} \psi\{pq\} &= \psi\left\{a_0b_0 - a_1b_1 + (a_1b_0 + a_0b_1)\vec{i} + (a_2b_0 + a_3b_1 + a_0b_2 - a_1b_3)\vec{j}\right. \\ &\qquad \left. + (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)\vec{k}\right\} \\ &= \psi(A + B\vec{i} + C\vec{j} + D\vec{k}) \\ &= \begin{bmatrix} A & -B & 0 & 0 \\ B & A & 0 & 0 \\ C & D & A & -B \\ D & -C & B & A \end{bmatrix} \\ &= \begin{bmatrix} a_0 & -a_1 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \otimes \begin{bmatrix} b_0 & -b_1 & 0 & 0 \\ b_1 & b_0 & 0 & 0 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{bmatrix} \\ &= \psi\{p\} \otimes \psi\{q\}. \end{aligned}$$

Thus the map ψ is a homomorphism. It is also one-to-one and onto and so ψ is an isomorphism.

If q is a nonzero semi-quaternion, the mapping

$$v_q : x \mapsto q x q^{-1},$$

is called the inner automorphism defined by q. We embed K into $\mathbb{R}_2^4$ by letting

$$x = (x_1, x_2, x_3) \mapsto (0, x_1, x_2, x_3) = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}.$$

The matrix representation of the map v_q is

$$M = \begin{bmatrix} a_0^2 + a_1^2 & 0 & 0 \\ 2(a_1a_2 - a_0a_3) & a_0^2 - a_1^2 & 2a_0a_1 \\ 2(a_1a_3 + a_0a_2) & -2a_0a_1 & a_0^2 - a_1^2 \end{bmatrix}.$$

Lemma 1: v_q is a linear mapping for all nonzero q, and it transforms the subspace of vectorial quaternions onto itself.

Proof: The linearity of $x \mapsto qx$ follows directly from Theorem 1. The argument $x \mapsto xp$ for is similar. Composition of these two mappings for $p = q^{-1}$ gives v_q ,so v_q is linear.

III. Results

According to definition 1, the kinematic mapping correspond with v_q is kinematic mapping of Blaschke and Grünwald. The corresponding geometry is not elliptic one, but so called quasi-elliptic geometry.

V. References

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