Horizontal lift in the semi-tensor bundle

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Abstract

The present paper is devoted to some results concerning with the horizontal lift of tensor fields of type (1,0) from manifold B to its semi-tensor (pull-back) bundle \(tB\) of type \((p,q)\). The Jacobian of (1.1) has components

\[
\begin{aligned}
\frac{\partial x^i}{\partial \tilde{x}^\alpha} = \alpha^i_{\alpha}, \\
\frac{\partial \tilde{x}^\alpha}{\partial x^i} = \alpha^i_{\alpha}.
\end{aligned}
\]

The present paper is devoted to some results concerning with the horizontal lift of tensor fields of type (1,0) from manifold B to its semi-tensor (pull-back) bundle \(tB\) of type \((p,q)\).

**Keywords:** Vector field, horizontal lift, pull-back bundle, semi-tensor bundle.

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1. Introduction

Let \(M_n\) be an \(n\)-dimensional differentiable manifold of class \(C^\infty\) and \(\pi_1 : M_n \to B_m\) the differentiable bundle determined by a submersion \(\pi_1\). Suppose that \((x') = (x^a, x'^a)\), \(a, b, \ldots = 1, \ldots, n = m; \alpha, \beta, \ldots = n - m + 1, \ldots, n\), \(i, j, \ldots = 1, 2, \ldots, n\) is a system of local coordinates adapted to the bundle \(\pi_1 : M_n \to B_m\), where \(x^a\) are coordinates in \(B_m\), and \(x'^a\) are fiber coordinates of the bundle \(\pi_1 : M_n \to B_m\).

\[
\begin{aligned}
\frac{\partial x^i}{\partial x'^j} = \alpha^i_{\alpha}, \\
\frac{\partial x'^j}{\partial x^i} = \alpha^i_{\alpha}.
\end{aligned}
\]

The Jacobian of (1.1) has components

\[
\begin{pmatrix}
A^i_{\alpha} & A^i_{\beta}
\end{pmatrix} = \begin{pmatrix}
A^i_{\alpha} & A^i_{\beta}
\end{pmatrix}.
\]

where

\[
\frac{\partial x'^a}{\partial x^i} = \alpha^a_{\alpha}.
\]

Let \((T^p_i)(B_m)(x = \pi_1(x), \tilde{x} = (x', x'^a) \in M_n)\) be the tensor space at a point \(x \in B_m\) with local coordinates \((x^1, \ldots, x^m)\), we have the holonomous frame field

\[
\partial_{x^1} \otimes \partial_{x^2} \otimes \cdots \otimes \partial_{x^i} \otimes dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k},
\]

for \(i \in \{1, \ldots, m\}\), \(j \in \{1, \ldots, m\}^q\), over \(U \subset B_m\) of this tensor bundle, and for any \((p,q)\)-tensor field \(t\) we have [14], p.163:

\[
t|U = t^{i_1\cdots i_q}_{j_1\cdots j_q} \partial_{x^{i_1}} \otimes \partial_{x^{i_2}} \otimes \cdots \otimes \partial_{x^{i_q}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \cdots \otimes dx^{j_k},
\]

then by definition the set of all points \((x') = (x^a, x'^a, x'^a)\), \(x^p = t^{i_1\cdots i_q}_{j_1\cdots j_q} x^i + m^p+q J, J, \ldots = 1, \ldots, n + m^p+q\) is a semi-tensor bundle \(T^p_i(B_m)\) over the manifold \(M_n\) [14]. The semi-tensor bundle \(T^p_i(B_m)\) has the natural bundle structure over \(B_m\), its bundle projection \(\pi : T^p_i(B_m) \to B_m\) being defined by \(\pi : (x^a, x'^a, x'^a) \to (x'^a)\). If we introduce a mapping \(\pi_2 : t^p_i(B_m) \to M_n\) by \(\pi_2 : (x^a, x'^a, x'^a) \to (x^a, x'^a)\), then \(t^p_i(B_m)\) has a bundle structure over \(M_n\). It is easily verified that \(\pi = \pi_1 \circ \pi_2\) [14].
On the other hand, let \( \varepsilon = \pi : E \to B \) denote a fiber bundle with fiber \( F \). Given a manifold \( B' \) and a map \( f : B' \to B \), one can construct in a natural way a bundle over \( B' \) with the same fiber: Consider the subset

\[
f^* E = \{ (b', \varepsilon) \in B' \times E \mid f(b') = \pi(\varepsilon) \}
\]

(together with the subspace topology from \( B' \times E \), and denote by \( \pi_1 : f^* E \to B' \), \( \pi_2 : f^* E \to E \) the projections. \( f^* \varepsilon = \pi_1 : f^* E \to B' \) is a fiber bundle with fiber \( F \), called the pull-back bundle of \( \varepsilon \) via \( f \) [31, 5, 8, 10, 14].

From the above definition it follows that the semi-tensor bundle \( t^0(B_m, \pi_2) \) is a pull-back bundle of the tensor bundle over \( B_m \) by \( \pi_1 \) (see, for example [12], [14]).

In other words, the semi-tensor bundle (induced or pull-back bundle) of the tensor bundle \( T^p(B_m, \pi, B_m) \) is the bundle \( t^p(B_m, \pi_2, M) \) over \( M \) with a total space \( t^p(B_m) \) \( = \{ ((x^a, x^{\alpha}), x^\Sigma) \in M \times (T^p(B_m)) : \forall (x^a, x^{\alpha}) = \overline{\pi}(x^a, x^\Sigma) = (x^a) \} \subset M \times (T^p(B_m)) \). To a transformation \( (1.1) \) of local coordinates of \( M \), there corresponds on \( t^p(B_m) \) the coordinate transformation

\[
\begin{pmatrix}
  x^a' \\
  x^{\alpha'} \\
  x^\Sigma'
\end{pmatrix} = \begin{pmatrix}
  A^{a'}_a & 0 & 0 \\
  0 & A^{\alpha'}_{\alpha} & 0 \\
  0 & 0 & A^{\Sigma'}_{\Sigma}
\end{pmatrix}
\begin{pmatrix}
  x^a \\
  x^{\alpha} \\
  x^\Sigma
\end{pmatrix},
\]

(1.2)

The Jacobian of (1.2) is given by [14]:

\[
\tilde{A} = (A^j_f) = \begin{pmatrix}
  A^f_{a'}^a & 0 & 0 \\
  0 & A^f_{\alpha'}^{\alpha} & 0 \\
  0 & 0 & A^{f}_{\Sigma'}^\Sigma
\end{pmatrix},
\]

(1.3)

where \( f = (a, \alpha, \Sigma) \), \( J = (b, \beta, \Phi) \), \( I, J, \ldots = 1, \ldots, n + m^p + q \), \( A^f_{a'}^a = A^a_{a'}, A^f_{\alpha'}^{\alpha} = A^{\alpha}_{\alpha'} \), \( A^{f}_{\Sigma'}^\Sigma = \frac{\partial x^\Sigma}{\partial x^{\Sigma'}} \).

It is easily verified that the condition \( \text{Det} \tilde{A} \neq 0 \) is equivalent to the condition:

\[
\text{Det}(A^j_f) \neq 0, \text{Det}(A^j_B) \neq 0, \text{Det}(A^j_{\alpha}) \neq 0.
\]

Also, \( \dim t^p(B_m) = n + m^p + q \). In the special case \( n = m \), \( t^p(B_m) \) is a tensor bundle \( T^p(B_m) \) [16], p.118. In the special case, the semi-tensor bundles \( t^1(B_m) \) \( (p = 1, q = 0) \) and \( t^0(B_m) \) \( (p = 0, q = 1) \) are semi-tangent and semi-cotangent bundles, respectively. We note that semi-tangent and semi-cotangent bundle were examined in \([11], [7], [9] \) and \([11], [13], [15], [16] \), respectively. Also, Fattaev studied the special class of semi-tensor bundle \([2] \). We denote by \( \Sigma^p_0(t^p(B_m)) \) and \( \Sigma^p_0(B_m) \) the modules over \( F(t^p(B_m)) \) and \( F(B_m) \) of all tensor fields of type \( (p, q) \) on \( t^p(B_m) \) and \( B_m \) respectively, where \( F(t^p(B_m)) \) and \( F(B_m) \) denote the rings of real-valued \( C^\infty \) – functions on \( t^p(B_m) \) and \( B_m \), respectively.

2. Horizontal lifts of vector fields and \( \gamma \) – Operator

Let \( \tilde{X} \in \Sigma^1_0(M_n) \) be a projectable vector field [9] with projection \( X = X^a(x^\alpha) \partial_a \) i.e. \( \tilde{X} = \tilde{X}^a(x^\alpha, x^{\alpha}) \partial_a + X^a(x^\alpha) \partial_a \). If we take account of (1.3), we can prove that \( \text{HH} \tilde{X} = \tilde{A}(\text{HH} X) \), where \( \text{HH} \tilde{X} \) is a vector field defined by

\[
\begin{align*}
\text{HH} \tilde{X} &= \begin{pmatrix}
  \tilde{X}^b \\
  \tilde{X}^{\beta} \\
  \tilde{X}^{(\gamma)} \end{pmatrix},
\end{align*}
\]

(2.1)

with respect to the coordinates \( (x^a, x^{\alpha}, \tilde{x}^\beta) \) on \( t^0(B_m) \). We call \( \text{HH} \tilde{X} \) the horizontal lift of the vector field of the vector field \( \tilde{X} \) to \( t^0(B_m) \) [14].

Now, consider \( A \in \Sigma^p_0(B_m) \) and \( \phi \in \Sigma^1_0(B_m) \), then \( v^\alpha A \in \Sigma^1_0(t^p(B_m)) \) \( (\text{vertical lift}) \), \( \gamma \phi \in \Sigma^1_0(t^p(B_m)) \) \( (\text{vertical lift}) \), and \( \gamma \phi \in \Sigma^1_0(t^p(B_m)) \) respectively, components on the semi-tensor bundle \( t^p(B_m) \) [14]

\[
v^\alpha A = \begin{pmatrix}
  0 \\
  0 \\
  A^a_{\beta \alpha} \partial_a \partial_{\beta} \partial_{\alpha}
\end{pmatrix}, \quad \gamma \phi = \begin{pmatrix}
  0 \\
  0 \\
  \gamma \phi^a_{\beta} \partial_a \partial_{\beta}
\end{pmatrix},
\]

(2.2)

\[
v^\alpha \gamma \phi \in \Sigma^1_0(B_m) \quad \text{on} \quad t^p(B_m) \quad \text{is defined by} \quad \gamma \phi = \varepsilon \circ \pi_2 = \varepsilon \circ \pi_1 \circ \pi_2 = \varepsilon \circ \pi.
\]

(2.3)

**Theorem 2.1.** For any vector fields \( \tilde{X}, \tilde{Y} \) on \( M_n \) and \( f \in \Sigma^0_0(B_m) \), we have

\[
\text{HH} \tilde{X} f = v^\alpha \epsilon (X f).
\]
Proof. Let $\tilde{X} \in \mathfrak{X}(M_\alpha)$. Then we get by (2.1) and (2.2):

\[
HH\tilde{X}^v f = HH\tilde{X}^\pi \partial_t (^v f)
\]

\[
HH\tilde{X}^v f &= HH\tilde{X}^a \partial_a (^v f) + HH\tilde{X}^\pi \partial_\pi (^v f) \\
&= X^a \partial_a (^v f) \\
&= ^v (X f),
\]

which gives Theorem 2.1.

\[\square\]

**Theorem 2.2.** Let $\tilde{X}$ be a projectable vector field on $M_\alpha$. For the Lie product, we have

\[
[HH\tilde{X}, ^v A] = ^v (V_X A)
\]

for any $A \in \mathfrak{X}_b^0 (B_m)$.

**Proof.** If $A, B \in \mathfrak{X}_b^0 (B_m)$ and \( \left[ HH\tilde{X}, ^v A \right]^b \) are components of \([HH\tilde{X}, ^v A]^l \) with respect to the coordinates \((x^b, x^\beta, x^\gamma)\) on \(t_b^0 (B_m)\), then we have

\[
[HH\tilde{X}, ^v A]^l = (HH\tilde{X})^l \partial_t (^v A)^l - (^v A)^l \partial_t (HH\tilde{X})^l
\]

\[
= (HH\tilde{X})^a \partial_a (^v A)^l + (HH\tilde{X})^\pi \partial_\pi (^v A)^l - (^v A)^a \partial_a (HH\tilde{X})^l - (HH\tilde{X})^\pi \partial_\pi (^v A)^l
\]

\[
= (HH\tilde{X})^a \partial_a (^v A)^l + (HH\tilde{X})^\pi \partial_\pi (^v A)^l + (^v A)^\pi \partial_\pi (HH\tilde{X})^l.
\]

Firstly, if $J = b$, we have

\[
[HH\tilde{X}, ^v A]^b = (HH\tilde{X})^a \partial_a (^v A)^b + (HH\tilde{X})^\pi \partial_\pi (^v A)^b
\]

\[
+ (HH\tilde{X})^\pi \partial_\pi (^v A)^b - (^v A)^\pi \partial_\pi (HH\tilde{X})^b
\]

\[
= \lambda_{pi} \partial_{pi} \tilde{X}^b
\]

\[
= 0,
\]

by virtue of (2.1) and (2.2). Secondly, if $J = \beta$, we have

\[
[HH\tilde{X}, ^v A]^{\beta} = (HH\tilde{X})^a \partial_a (^v A)^{\beta} + (HH\tilde{X})^\pi \partial_\pi (^v A)^{\beta}
\]

\[
+ (HH\tilde{X})^\pi \partial_\pi (^v A)^{\beta} - (^v A)^\pi \partial_\pi (HH\tilde{X})^{\beta}
\]

\[
= \lambda_{pi} \partial_{pi} \tilde{X}^{\beta}
\]

\[
= 0,
\]
by virtue of (2.1) and (2.2). Thirdly, if $J = \overline{\beta}$, then we have

$$
[\overline{\mathcal{H}} X, X \mathcal{V} A^\overline{\beta}] = \left( \mathcal{H} \overline{X} \right)^\alpha \partial_\alpha (X^\mathcal{V} A^\overline{\beta}) + \left( \mathcal{H} \overline{X} \right)^\alpha \partial_\alpha (X^\mathcal{V} A^\overline{\beta})
$$

by virtue of (2.1) and (2.2). On the other hand, we know that $\mathcal{V} \overline{\partial}_X$ have components

$$
\mathcal{V} \overline{\partial}_X = \begin{pmatrix} 0 & \ldots & 0 \\ (\nabla X A) & \ldots & (\nabla X A) \end{pmatrix}
$$

with respect to the coordinates $X^\mathcal{V} \overline{\partial}_X$ on $t_{\mathcal{V}}(B_m)$. Thus Theorem 2.2 is proved.

We denote the curvature tensor of $\nabla$ by $R \in \mathfrak{S}(B_m)$. Then $R(X, Y)$ is an element of $\mathfrak{S}(B_m)$ such that,

$$
R(X, Y)Z = [\nabla X, \nabla Y]Z - \nabla [X, Y]Z
$$

for any $X, Y, Z \in \mathfrak{S}(B_m)$.

From (2.1) we have:

**Theorem 2.3.** Let $\tilde{X}$ and $\tilde{Y}$ be projectable vector fields on $M_n$ with projections $X$ and $Y$ on $B_m$, respectively. For the Lie product, we have

$$
[\mathcal{H} \tilde{X}, \mathcal{H} \tilde{Y}]^\mathcal{V} = \mathcal{H} [\tilde{X}, \tilde{Y}]^\mathcal{V} + \mathcal{V} (\tilde{X} - \tilde{Y}) R(X, Y).
$$

**Proof.** If $\tilde{X}$ and $\tilde{Y}$ are projectable vector fields on $M_n$ with projection $X, Y \in \mathfrak{S}(B_m)$ and

$$
\begin{pmatrix} [\mathcal{H} \tilde{X}, \mathcal{H} \tilde{Y}]^\mathcal{V} \\ [\mathcal{H} \tilde{X}, \mathcal{H} \tilde{Y}]^\beta \end{pmatrix}
$$

are components of $[\mathcal{H} \tilde{X}, \mathcal{H} \tilde{Y}]^\mathcal{V}$ with respect to the coordinates $X^\mathcal{V} \overline{\partial}_X$ on $t_{\mathcal{V}}(B_m)$, then we have

$$
[\mathcal{H} \tilde{X}, \mathcal{H} \tilde{Y}]^\mathcal{V} = [\mathcal{H} \tilde{X}]^\mathcal{V} \partial_X (\mathcal{H} \tilde{Y})^\mathcal{V} - ([\mathcal{H} \tilde{Y}]^\mathcal{V} \partial_X (\mathcal{H} \tilde{X})^\mathcal{V}.
$$

Firstly, if $J = b$, we have

$$
[\mathcal{H} \tilde{X}, \mathcal{H} \tilde{Y}]^b = \mathcal{H} \tilde{X}^\gamma \partial_\gamma (\mathcal{H} \tilde{Y})^b - \mathcal{H} \tilde{Y}^\gamma \partial_\gamma (\mathcal{H} \tilde{X})^b + \mathcal{H} \tilde{X}^\gamma \partial_\gamma (\mathcal{H} \tilde{Y})^b + \mathcal{H} \tilde{Y}^\gamma \partial_\gamma (\mathcal{H} \tilde{X})^b.
$$

Finally, if $J = b$, we have

$$
[\mathcal{H} \tilde{X}, \mathcal{H} \tilde{Y}]^b = \mathcal{H} \tilde{X}^\gamma \partial_\gamma (\mathcal{H} \tilde{Y})^b - \mathcal{H} \tilde{Y}^\gamma \partial_\gamma (\mathcal{H} \tilde{X})^b + \mathcal{H} \tilde{X}^\gamma \partial_\gamma (\mathcal{H} \tilde{Y})^b + \mathcal{H} \tilde{Y}^\gamma \partial_\gamma (\mathcal{H} \tilde{X})^b.
$$

Thus Theorem 2.2 is proved.

\[\square\]
by virtue of (2.1). Secondly, if $J = \beta$, we have

\[
\begin{align*}
\{HH\bar{X}, HH\bar{Y}\}^\beta &= (HH\bar{X})^a \partial_a (HH\bar{Y})^\beta - (HH\bar{Y})^a \partial_a (HH\bar{X})^\beta \\
&= (HH\bar{X})^a \partial_a (HH\bar{Y})^\beta - (HH\bar{Y})^a \partial_a (HH\bar{X})^\beta - (HH\bar{Y})^a \partial_a (HH\bar{Y})^\beta - (HH\bar{Y})^a \partial_a (HH\bar{X})^\beta \\
&= (HH\bar{X})^a \partial_a (HH\bar{Y})^\beta - (HH\bar{Y})^a \partial_a (HH\bar{X})^\beta \\
&= X^a \partial_a \bar{Y}^\beta - Y^a \partial_a \bar{X}^\beta \\
&= [X, Y]^\beta
\end{align*}
\]

by virtue of (2.1). Thirdly, if $J = \beta$, we have

\[
\begin{align*}
\{HH\bar{X}, HH\bar{Y}\}^\beta &= HH\bar{X}^a \partial_a (HH\bar{Y})^\beta - HH\bar{Y}^a \partial_a (HH\bar{X})^\beta \\
&= HH\bar{X}^a \partial_a (HH\bar{Y})^\beta + HH\bar{X}^a \partial_a HH\bar{Y}^\beta + HH\bar{X}^a \partial_a HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta \\
&= X^a \partial_a HH\bar{Y}^\beta - \sum_{\lambda=1}^{p} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{Y}^\beta \\
&+ \sum_{\mu=1}^{q} \sum_{\beta_1, \ldots, \beta_q} \sum_{\gamma} \alpha_1 \ldots \alpha_q \Gamma_{\beta_1 \ldots \beta_q}^\gamma \epsilon_{\beta_1 \ldots \beta_q} \partial_{\gamma} HH\bar{X}^\beta
by virtue of (2.1). On the other hand, we know that $HH [X, Y] + (\gamma - \gamma) R(X, Y)$ have components

$$
HH [X, Y] + (\gamma - \gamma) R(X, Y)
$$

with respect to the coordinates $(x^b, \gamma^b, \gamma^c)$ on $\iota_\gamma^n(B_m)$. Thus Theorem 2.3 is proved.
References