

## SOME SUMMATION FORMULAS FOR THE HYPERGEOMETRIC SERIES ${}_{r+2}F_{r+1}\left(\frac{1}{2}\right)$

Y. S. Kim\*, A. K. Rathie†, U. Pandey‡ and R. B. Paris§ ¶

Received 16:06:2011 : Accepted 22:06:2012

### Abstract

The aim of this paper is to obtain explicit expressions of the generalized hypergeometric function

$${}_{r+2}F_{r+1}\left[\begin{matrix} a, b, \\ \frac{1}{2}(a+b+j+1), \end{matrix} \begin{matrix} (f_r+m_r) \\ (f_r) \end{matrix}; \frac{1}{2}\right]$$

for  $j = 0, \pm 1, \dots, \pm 5$ , where  $r$  pairs of numeratorial and denominatorial parameters differ by positive integers  $m_r$ . The results are derived with the help of an expansion in terms of a finite sum of  ${}_2F_1\left(\frac{1}{2}\right)$  functions and a generalization of Gauss' second summation theorem due to Lavoie *et al.* [J. Comput. Appl. Math. **72**, 293–300 (1996)]. Some special and limiting cases are also given.

**Keywords:** Generalized hypergeometric series, Generalized Gauss summation theorem

*2000 AMS Classification:* Primary 33C20, Secondary 33C15

---

\*Department of Mathematics Education, Wonkwang University, Iksan, Korea.

E-Mail: yspkim@wonkwang.ac.kr

†Department of Mathematics, Central University of Kerala, Kasaragad 671328, Kerala, India.

E-Mail: akrathie@rediffmail.com

‡Department of Mathematics, Marudhar Engineering College, Raisar, Bikaner, Rajasthan State, India. E-mail: pandey.up@rediffmail.com

§University of Abertay Dundee, Dundee DD1 1HG, UK. E-Mail: r.paris@abertay.ac.uk

¶Corresponding author

### 1. Introduction

The generalized hypergeometric function with  $p$  numeratorial and  $q$  denominatorial parameters is defined by the series [16, p. 41]

$$(1.1) \quad {}_pF_q \left[ \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} ; x \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{x^n}{n!},$$

where  $(a)_n = \Gamma(a + n)/\Gamma(a)$  is the Pochhammer symbol (or ascending factorial). When  $q = p$  this series converges for  $|x| < \infty$ , but when  $q = p - 1$  convergence occurs when  $|x| < 1$  (unless the series terminates). In what follows we shall adopt the convention of writing the finite sequence of parameters  $(a_1, \dots, a_p)$  simply by  $(a_p)$ .

It is well known that whenever hypergeometric functions reduce to gamma functions, the results are very important from the applications point of view. Thus, the classical theorems of Gauss, Kummer and Bailey for the series  ${}_2F_1$ , and of Watson, Dixon, Whipple and Saalschütz for the series  ${}_3F_2$ , and others, play an important role in the theory of hypergeometric and generalized hypergeometric series. In [3, 4, 5], Lavoie *et al.* considered generalizations of some of the above-mentioned classical summations. In particular, they obtained a generalization of Gauss' second summation theorem, given by

$${}_2F_1 \left[ \begin{matrix} a, b \\ \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \end{matrix} ; \frac{1}{2} \right] = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})}$$

provided  $\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \neq 0, -1, -2, \dots$ , in the following form

**1.1. Theorem.** [5] *Provided  $\frac{1}{2}(a + b + j + 1) \neq 0, -1, -2, \dots$ , we have the summation*

$$(1.2) \quad {}_2F_1 \left[ \begin{matrix} a, b \\ \frac{1}{2}(a + b + j + 1) \end{matrix} ; \frac{1}{2} \right] = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}j + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}j + \frac{1}{2})}{\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2} + \frac{1}{2}|j|)} \\ \times \left\{ \frac{A_j(a, b)}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \delta_{j+1})} + \frac{B_j(a, b)}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b + \delta_j)} \right\}$$

for integer  $j$ . As usual,  $[x]$  denotes the greatest integer less than or equal to  $x$ , its modulus is denoted by  $|x|$  and we have defined

$$(1.3) \quad \delta_j \equiv \frac{1}{2}j - [\frac{1}{2}j].$$

The coefficients  $A_j(a, b)$  and  $B_j(a, b)$  are displayed in Table 1 for  $0 \leq j \leq 5$ , where the coefficients for  $j < 0$  satisfy

$$(1.4) \quad A_{-j}(a, b) = (-1)^j A_j(a, b), \quad B_{-j}(a, b) = (-1)^{j+1} B_j(a, b).$$

TABLE 1. The coefficients  $A_j$  and  $B_j$  for  $0 \leq j \leq 5$ .

$j$	$A_j(a, b)$	$B_j(a, b)$
0	1	0
1	-1	1
2	$\frac{1}{2}(a + b - 1)$	-2
3	$-\frac{1}{2}(3a + b - 2)$	$\frac{1}{2}(3b + a - 2)$
4	$\frac{1}{4}(a + b - 1)(a + b - 3) + ab$	$-2(a + b - 1)$
5	$-\frac{1}{4}(a + b - 2)(5a + b - 4) + a(b + 1)$	$\frac{1}{4}(a + b - 2)(5b + a - 4) + b(a + 1)$

Recently Miller [7], Miller and Paris [8, 9] and Miller and Srivastava [12] studied the generalized hypergeometric series (including integer parameter differences) and obtained numerous transformation and summation formulas for the series  ${}_{r+2}F_{r+1}(z)$  and  ${}_{r+1}F_{r+1}(z)$ . In our present investigation, we shall require the following result established in [8] expressing  ${}_{r+2}F_{r+1}(z)$ , with  $r$  numeratorial and denominatorial parameters differing by the positive integers  $(m_r)$ , in terms of a finite sum of Gauss hypergeometric functions.

**1.2. Theorem.** [8] *Let  $(m_r)$  denote a set of positive integers with  $m = m_1 + \dots + m_r$ . Then, when  $|z| < 1$ , we have*

$$(1.5) \quad {}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, (f_r + m_r) \\ c, (f_r) \end{matrix} ; z \right] = \sum_{k=0}^m \frac{(a)_k (b)_k}{(c)_k} z^k C_k(r) {}_2F_1 \left[ \begin{matrix} a+k, b+k \\ c+k \end{matrix} ; z \right],$$

where the coefficients  $C_k(r)$  are defined by [10]

$$(1.6) \quad C_k(r) = \frac{(-1)^k}{k!} {}_{r+1}F_r \left[ \begin{matrix} -k, (f_r + m_r) \\ (f_r) \end{matrix} ; 1 \right].$$

The expansion also holds when  $z = 1$  provided  $\text{Re}(c - a - b) > m$ .

Alternatively, the coefficients can be expressed in the form [8]

$$(1.7) \quad C_k(r) \equiv \frac{1}{\Lambda} \sum_{j=k}^m \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \sigma_{m-j}, \quad \Lambda = (f_1)_{m_1} \dots (f_r)_{m_r},$$

with  $C_0(r) = 1$ ,  $C_m(r) = 1/\Lambda$ , where  $\left\{ \begin{matrix} j \\ k \end{matrix} \right\}$  denotes the Stirling number of the second kind and the  $\sigma_j$  ( $0 \leq j \leq m$ ) are generated by the relation

$$(1.8) \quad (f_1 + x)_{m_1} \dots (f_r + x)_{m_r} = \sum_{j=0}^m \sigma_{m-j} x^j.$$

The result (1.5) and (1.6) can also be deduced as a particular case of the more general expansion given by Luke in [6, Eq. (5.10.2(4))] combined with the fact that  $C_k(r) = 0$  for  $k > m$  [2, 10].

**1.3. Remark.** If we set  $z = 1$  in (1.5), we immediately obtain the generalization of the Karlsson-Minton summation theorem [11, 12]

$${}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, (f_r + m_r) \\ c, (f_r) \end{matrix} ; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^m \frac{(-)^k (a)_k (b)_k C_k(r)}{(1+a+b-c)_k}$$

provided  $\text{Re}(c-a-b) > m$ . If  $c = b+1$  and use is made of the alternative representation of the coefficients  $C_k(r)$  in (1.7), this reduces to the Karlsson-Minton summation formula given by [11, 13]

$${}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, (f_r + m_r) \\ b+1, (f_r) \end{matrix} ; 1 \right] = \frac{\Gamma(1+b)\Gamma(1-a)}{\Gamma(1+b-a)} \frac{(f_1-b)_{m_1} \dots (f_r-b)_{m_r}}{(f_1)_{m_1} \dots (f_r)_{m_r}}$$

when  $\text{Re}(-a) > m-1$ .

Our aim in this paper is to obtain explicit expressions of

$${}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, (f_r + m_r) \\ \frac{1}{2}(a+b+j+1), (f_r) \end{matrix} ; \frac{1}{2} \right]$$

for complex parameters  $a, b$  and  $(f_r)$  and for  $j = 0, \pm 1, \dots, \pm 5$ , where  $r$  pairs of numeratorial and denominatorial parameters differ by positive integers  $(m_r)$ . For this purpose we shall make use of the expansion (1.5) combined with the generalization in (1.2) of Gauss'

second summation theorem. Several special and limiting cases of our main findings are also presented.

### 2. Summation formulas

Our main result is given by the following theorem.

**2.1. Theorem.** *Let  $(m_r)$  be a set of positive integers with  $m = m_1 + \dots + m_r$  and let  $j$  be an integer. Then, the generalized hypergeometric function of argument  $\frac{1}{2}$  and with  $r$  pairs of numeratorial and denominatorial parameters differing by positive integers has the summation*

$$\begin{aligned}
 (2.1) \quad & {}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}(a + b + j + 1), & (f_r) \end{matrix} ; \frac{1}{2} \right] \\
 &= \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}j + \frac{1}{2})\Gamma(\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}j + \frac{1}{2})}{\Gamma(\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}|j| + \frac{1}{2})} \sum_{k=0}^m (a)_k (b)_k C_k(r) \left(\frac{1}{2}\right)^k \\
 &\times \left\{ \frac{A_j(a + k, b + k)}{\Gamma(\frac{1}{2}a + \frac{1}{2}k + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2}k + \delta_{j+1})} + \frac{B_j(a + k, b + k)}{\Gamma(\frac{1}{2}a + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2}k + \delta_j)} \right\},
 \end{aligned}$$

where the coefficients  $A_j$  and  $B_j$  for  $-5 \leq j \leq 5$  can be obtained from the  $A_j(a, b)$  and  $B_j(a, b)$  in Table 1 and use of (1.4) simply by putting  $a \mapsto a + k$  and  $b \mapsto b + k$ , and  $\delta_j$  is defined in (1.3).

*Proof.* If we set  $z = \frac{1}{2}$  and  $c = \frac{1}{2}(a + b + j + 1)$ , where  $j$  is an integer, in (1.5) we obtain

$$\begin{aligned}
 & {}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}(a + b + j + 1), & (f_r) \end{matrix} ; \frac{1}{2} \right] \\
 &= \sum_{k=0}^m \frac{(a)_k (b)_k C_k(r)}{2^k (\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}j + \frac{1}{2})_k} {}_2F_1 \left[ \begin{matrix} a + k, b + k \\ \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}j + \frac{1}{2} + k \end{matrix} ; \frac{1}{2} \right].
 \end{aligned}$$

Application of (1.2) then leads to the result stated in (2.1). □

To simplify the presentation of specific cases, we define the quantities  $D_{\pm}^{(1)}(k)$  and  $D_{\pm}^{(2)}(k)$  by

$$\begin{aligned}
 D_{\pm}^{(1)}(k) &\equiv \frac{1}{a \pm b} \left( \Gamma(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2}k) \pm \Gamma(\frac{1}{2}a + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2} + \frac{1}{2}k) \right), \\
 D_{\pm}^{(2)}(k) &\equiv \frac{1}{(a \pm b)^2 - 1} \left( (a + b + 2k - 1)\Gamma(\frac{1}{2}a + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2}k) \right. \\
 &\quad \left. \pm 4\Gamma(\frac{1}{2}a + \frac{1}{2} + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2} + \frac{1}{2}k) \right).
 \end{aligned}$$

Then, upon use of the duplication formula for the gamma function

$$(2.2) \quad \sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z)\Gamma(z + \frac{1}{2}),$$

we have the following summations.

**2.2. Corollary.** *The summations in (2.1) for  $j = 0, \pm 1, \pm 2$ , respectively, take the form*

$$\begin{aligned}
 (2.3) \quad & {}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, & (f_r + m_r) \\ \frac{1}{2}(a + b + 1), & (f_r) \end{matrix} ; \frac{1}{2} \right] \\
 &= \frac{2^{a+b-2}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(a)\Gamma(b)} \sum_{k=0}^m 2^k C_k(r) \Gamma(\frac{1}{2}a + \frac{1}{2}k)\Gamma(\frac{1}{2}b + \frac{1}{2}k),
 \end{aligned}$$

(2.4)

$$\begin{aligned} & {}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, \\ \frac{1}{2}(a+b+1) \pm \frac{1}{2}, \end{matrix} \begin{matrix} (f_r + m_r) \\ (f_r) \end{matrix} ; \frac{1}{2} \right] \\ &= \frac{2^{a+b-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + 1)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^m 2^k C_k(r) D_{\mp}^{(1)}(k), \end{aligned}$$

(2.5)

$$\begin{aligned} & {}_{r+2}F_{r+1} \left[ \begin{matrix} a, b, \\ \frac{1}{2}(a+b+1) \pm 1, \end{matrix} \begin{matrix} (f_r + m_r) \\ (f_r) \end{matrix} ; \frac{1}{2} \right] \\ &= \frac{2^{a+b-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{3}{2})}{\Gamma(a)\Gamma(b)} \sum_{k=0}^m 2^k C_k(r) D_{\mp}^{(2)}(k). \end{aligned}$$

When  $r = 0$ , the sequences  $(f_r)$  and  $(f_r + m_r)$  are empty so that  $m = 0$  and (2.3)–(2.5) reduce to (1.2).

**2.3. Remark.** The results for  $j = -2, -1, 0$  and  $1$  have been obtained by Miller and Paris [8]. However, the results for  $j = -2$  and  $j = -1$  are given here in corrected forms.

### 3. Specific examples

As specific examples, we let  $r = 1, m_1 = 1$  and  $f_1 = f$ , so that  $C_0(1) = 1, C_1(1) = 1/f$ , and thus obtain

$$\begin{aligned} (3.1) \quad & {}_3F_2 \left[ \begin{matrix} a, b, \\ \frac{1}{2}(a+b+1) \end{matrix} \begin{matrix} f+1 \\ f \end{matrix} ; \frac{1}{2} \right] \\ &= \pi^{\frac{1}{2}} \Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}) \left\{ \frac{1}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})} + \frac{2/f}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)} \right\}, \end{aligned}$$

$$\begin{aligned} (3.2) \quad & {}_3F_2 \left[ \begin{matrix} a, b, \\ \frac{1}{2}(a+b+1) \pm \frac{1}{2}, \end{matrix} \begin{matrix} f+1 \\ f \end{matrix} ; \frac{1}{2} \right] \\ &= 2\pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + 1)}{a \mp b} \left\{ \frac{(a \mp f)/f}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b)} \mp \frac{(b \mp f)/f}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b + \frac{1}{2})} \right\}, \end{aligned}$$

$$\begin{aligned} (3.3) \quad & {}_3F_2 \left[ \begin{matrix} a, b, \\ \frac{1}{2}(a+b+1) \pm 1, \end{matrix} \begin{matrix} f+1 \\ f \end{matrix} ; \frac{1}{2} \right] \\ &= 2\pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{3}{2})}{(a \mp b)^2 - 1} \left\{ \frac{a+b-1 \mp (2ab)/f}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})} \mp \frac{4 \mp 2(a+b+1)/f}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)} \right\}. \end{aligned}$$

**3.1. Remark.** The summation in (3.2) with the upper signs has been obtained previously in [10] and [15].

The summations in (2.1) corresponding to  $j \geq 1$  are not valid when  $a-b = 0, \pm 1, \dots, \pm(\frac{1}{2}j - \frac{1}{2})$  for odd  $j$ , and when  $a-b = \pm\frac{1}{2}, \dots, \pm(\frac{1}{2}j - \frac{1}{2})$  for even  $j$ . In such cases we may use l'Hôpital's rule and the duplication formula (2.2) to obtain the limiting cases given below.

**3.2. Corollary.** When  $a = b$ , for example, we find the following summations corresponding to  $j = 1, 3$  and  $5$  respectively:

$$(3.4) \quad {}_{r+2}F_{r+1} \left[ \begin{matrix} a, a, (f_r + m_r) \\ a + 1, (f_r) \end{matrix}; \frac{1}{2} \right]$$

$$(3.5) \quad = 2^{a-1} a \sum_{k=0}^m C_k(r)(a)_k \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}k + \frac{1}{2}\right) - \psi\left(\frac{1}{2}a + \frac{1}{2}k\right) \right\},$$

$$(3.6) \quad {}_{r+2}F_{r+1} \left[ \begin{matrix} a, a, (f_r + m_r) \\ a + 2, (f_r) \end{matrix}; \frac{1}{2} \right] = 2^{a-1} a(a+1) \sum_{k=0}^m C_k(r)(a)_k \\ \left\{ (2a + 2k - 1) \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}k\right) - \psi\left(\frac{1}{2}a + \frac{1}{2}k + \frac{1}{2}\right) \right\} + 2 \right\}$$

and

$${}_{r+2}F_{r+1} \left[ \begin{matrix} a, a, (f_r + m_r) \\ a + 3, (f_r) \end{matrix}; \frac{1}{2} \right] = 2^{a-1} a(a+1)(a+2) \sum_{k=0}^m C_k(r)(a)_k \\ \left\{ \left( (a+k - \frac{1}{2})^2 + \frac{1}{4} \right) \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}k + \frac{1}{2}\right) - \psi\left(\frac{1}{2}a + \frac{1}{2}k\right) \right\} - (a+k - \frac{1}{2}) \right\},$$

where  $\psi$  is the digamma or psi function.

**3.3. Corollary.** Similarly, if  $a - b = 1$  when  $j = 2$  and  $4$  we find the summations:

$$(3.7) \quad {}_{r+2}F_{r+1} \left[ \begin{matrix} a, a + 1, (f_r + m_r) \\ a + 2, (f_r) \end{matrix}; \frac{1}{2} \right] \\ = 2^a a(a+1) \sum_{k=0}^m C_k(r)(a+1)_k \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}k + \frac{1}{2}\right) - \psi\left(\frac{1}{2}a + \frac{1}{2}k\right) - \frac{1}{a+k} \right\}$$

and

$$(3.8) \quad {}_{r+2}F_{r+1} \left[ \begin{matrix} a, a + 1, (f_r + m_r) \\ a + 3, (f_r) \end{matrix}; \frac{1}{2} \right] = 2^a a(a+1)(a+2) \sum_{k=0}^m C_k(r)(a+1)_k \\ \left\{ (a+k) \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}k\right) - \psi\left(\frac{1}{2}a + \frac{1}{2}k + \frac{1}{2}\right) \right\} + 1 + \frac{1}{2(a+k)} \right\}.$$

Finally, it is worthy of mention that when  $r = 0$  we retrieve respectively from (3.4)–(3.8), or from (1.2), the following results for the series  ${}_2F_1(\frac{1}{2})$ :

$${}_2F_1 \left[ \begin{matrix} a, a \\ a + 1 \end{matrix}; \frac{1}{2} \right] = 2^{a-1} a \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}\right) - \psi\left(\frac{1}{2}a\right) \right\},$$

$${}_2F_1 \left[ \begin{matrix} a, a \\ a + 2 \end{matrix}; \frac{1}{2} \right] = 2^{a-1} a(a+1) \left\{ (2a-1) \left\{ \psi\left(\frac{1}{2}a\right) - \psi\left(\frac{1}{2}a + \frac{1}{2}\right) \right\} + 2 \right\},$$

$${}_2F_1 \left[ \begin{matrix} a, a \\ a + 3 \end{matrix}; \frac{1}{2} \right] = 2^{a-1} a(a+1)(a+2) \\ \left\{ \left( (a - \frac{1}{2})^2 + \frac{1}{4} \right) \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}\right) - \psi\left(\frac{1}{2}a\right) \right\} - (a - \frac{1}{2}) \right\},$$

$${}_2F_1 \left[ \begin{matrix} a, a + 1 \\ a + 2 \end{matrix}; \frac{1}{2} \right] = 2^a a(a+1) \left\{ \psi\left(\frac{1}{2}a + \frac{1}{2}\right) - \psi\left(\frac{1}{2}a\right) - \frac{1}{a} \right\}$$

and

$${}_2F_1 \left[ \begin{matrix} a, a+1 \\ a+3 \end{matrix}; \frac{1}{2} \right] = 2^a a(a+1)(a+2) \left\{ a \left\{ \psi\left(\frac{1}{2}a\right) - \psi\left(\frac{1}{2}a + \frac{1}{2}\right) \right\} + 1 + \frac{1}{2a} \right\}.$$

**3.4. Remark.** The first of the above  ${}_2F_1(\frac{1}{2})$  summations is found in [1, p. 557] and in [14, p. 492] written in slightly different form, but the remaining results are believed to be new.

## Acknowledgments

One of the authors (YSK) acknowledges the support of the Wonkwang University Research Fund (2012).

## References

- [1] Abramowitz, M. and Stegun, I. A. (Eds.) *Handbook of Mathematical Functions* (Dover, New York, 1965).
- [2] Karlsson, P. W. *Hypergeometric functions with integral parameter differences*, J. Math. Phys. **12**, 270–271, 1971.
- [3] Lavoie, J. L., Grondin, F. and Rathie, A. K. *Generalizations of Watson's theorem on the sum of a  ${}_3F_2$* , Indian J. Math. **34**, 23–32, 1992.
- [4] Lavoie, J. L., Grondin, F. and Rathie, A. K. *Generalizations of Dixon's theorem on the sum of a  ${}_3F_2$* , Math. Comp. **62**, 267–276, 1994.
- [5] Lavoie, J. L., Grondin, F. and Rathie, A. K. *Generalizations of Whipple's theorem on the sum of a  ${}_3F_2$* , J. Comput. Appl. Math. **72**, 293–300, 1996.
- [6] Luke, Y. L. *Mathematical Functions and Their Approximations* (Academic Press, New York, 1975).
- [7] Miller, A. R. *Certain summation and transformation formulas for generalized hypergeometric series*, J. Comput. Appl. Math. **231**, 964–972, 2009.
- [8] Miller, A. R. and Paris, R. B. *Certain transformations and summations for the generalized hypergeometric series with integral parameter differences*, Integral Transforms and Special Functions, **22**, 67–77, 2011.
- [9] Miller, A. R. and Paris, R. B. *Euler-type transformations for the generalized hypergeometric function  ${}_{r+2}F_{r+1}(x)$* , Zeitschrift angew. Math. Phys., **62**, 31–45, 2011.
- [10] Miller, A. R. and Paris, R. B. *On a result related to transformations and summations of generalized hypergeometric series*, Math. Communications, **17**, 205–210, 2012.
- [11] Miller, A. R. and Paris, R. B. *Transformation formulas for the generalized hypergeometric function with integral parameter differences*, Rocky Mountain J. Math., **43**, 291–327, 2013.
- [12] Miller, A. R. and Srivastava, H. M. *Karlsson–Minton summation theorems for the generalized hypergeometric series of unit argument*, Integral Transforms and Special Functions, **21**, 603–612, 2010.
- [13] Minton, B. M. *Generalized hypergeometric function of unit argument*, J. Math. Phys., **11**, 1375–1376, 1970.
- [14] Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I. *Integrals and Series: More Special Functions*, vol. 3 (Gordon and Breach, New York, 1990).
- [15] Rathie, A. K. and Pogány, T. K. *New summation formula for  ${}_3F_2(\frac{1}{2})$  and a Kummer-type II transformation of  ${}_2F_2(x)$* , Mathematical Communications, **13**, 63–66, 2008.
- [16] Slater, L. J. *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge, 1966).