

ON THE SEMI-MARKOVIAN RANDOM WALK WITH DELAY AND WEIBULL DISTRIBUTED INTERFERENCE OF CHANCE

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Abstract

In this paper, a semi-Markovian random walk with delay and a discrete interference of chance ($X(t)$) is considered. It is assumed that the random variables $\{\zeta_n\}$, $n \geq 1$ which describe the discrete interference of chance have Weibull distribution with parameters (α, λ) , $\alpha > 1$, $\lambda > 0$. Under this assumption, the ergodicity of this process is discussed and the asymptotic expansions with three terms for the first four moments of the ergodic distribution of the process $X(t)$ are derived, when $\lambda \rightarrow 0$. Moreover, the asymptotic expansions for the skewness and kurtosis of the ergodic distribution of the process $X(t)$ are established.

Keywords: Semi-Markovian random walk; a discrete interference of chance; Weibull distribution; ergodic distribution; asymptotic expansion; ladder variables.

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1. Introduction

Many applied problems of the queueing, reliability, inventory control, insurance and other theories are formulated in the terms of random walks with various types of barrier. Some important studies on this topic exist in the literature (see, for example, [1–9]). Let us consider the following model before stating the problem mathematically.

The Model. Suppose that, the system is in state $z = s + x$ at the initial time $t = 0$. Here, $s > 0$ is a predefined control level, and $x > 0$. Demands and supplies are occurred at the random times $T_n = \sum_{i=1}^n \xi_i$, $n \geq 1$. System passes from a state to another one by jumping at time T_n , according to quantities of demands and supplies $\{\eta_n\}$, $n \geq 1$. This change of system continues until certain random time τ_1 , where τ_1 is the first passage time

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to the control level s . When this case happens, by interfering to the system from external, systems is stopped at the level s for a random time θ_1 . Usually the random variables θ_1 and $K = E\theta_1/E\xi_1$ are called as delaying time and delaying coefficient for the system, respectively. Then, as a consequence of external interference, system is brought from the control level s to state ζ_1 . Thus, the first period has been completed. Afterwards, system will continue its function similar to the preceding period.

Note that, in the study [8], [2] and [1] the random variable ζ_1 , which describes the discrete interference of chance, has an exponential, triangular and gamma distribution, respectively and the stationary moments of ergodic distribution were investigated when the delaying time is zero, i.e. $\theta_1 = 0$. But the delaying time is necessary for many real systems. So, in this study, unlike [8], [2] and [1], the asymptotic expansions with three terms for the first four moments of the ergodic distribution of the process $X(t)$ will be investigated by taking into account the delaying time (θ_1). Also in this study we assume that the random variable ζ_1 has Weibull distribution with parameters (α, λ) , $\alpha > 1$, $\lambda > 0$.

Our aim, in this paper, is to investigate the asymptotic behavior of the ergodic moments of this process $t \in [\gamma_n, \tau_{n+1})$, when $\lambda \rightarrow 0$.

2. Mathematical construction of the process $X(t)$

Let $\{(\xi_n, \eta_n, \theta_n, \zeta_n)\}$, $n \geq 1$ be a sequence of independent and identically distributed vector of random variables defined on any probability space $(\Omega, \mathfrak{F}, P)$, such that ξ_n and θ_n take only positive values, η_n takes negative values as well as positive ones; ζ_n has Weibull distribution with parameters (α, λ) , $\alpha > 1$, $\lambda > 0$. Suppose that $\xi_1, \eta_n, \theta_1, \zeta_1$ are mutually independent random variables and the distribution functions of them are known, i.e.,

$$\begin{aligned} \Phi(t) &= P\{\xi_1 \leq t\}; F(x) = P\{\eta_1 \leq x\}; H(u) = P\{\theta_1 \leq u\}; t \geq 0; \\ x \in (-\infty, +\infty) u \geq 0 \text{ and } \pi(z) &= P\{\zeta_1 \leq z\} = 1 - e^{-(\lambda z)^\alpha}, z \geq 0, \alpha > 1, \lambda > 0. \end{aligned}$$

Define renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ as follows:

$$T_n = \sum_{i=1}^n \xi_i, S_n = \sum_{i=1}^n \eta_i, T_0 = S_0 = 0, n = 1, 2, \dots$$

and a sequence of integer valued random variables $\{N_n\}$ as:

$$\begin{aligned} N_0 &= 0, \\ N_1 &= \inf\{n \geq 1 : s + x - S_n < s\} = \inf\{n \geq 1 : S_n > x\} = N(x), x \geq 0; \\ N_{n+1} &= \inf\left\{k \geq 1 : s + \zeta_n - \left(\sum_{i=N_1+N_2+\dots+N_n+1}^{N_1+N_2+\dots+N_n+k} \eta_i\right) < s\right\} \\ &= \inf\{k \geq 1 : S_{N_1+N_2+\dots+N_n+k} - S_{N_1+N_2+\dots+N_n} > \zeta_n\}, n = 1, 2, \dots \end{aligned}$$

Here $s > 0$ and $\inf\{\emptyset\} = +\infty$ is stipulated.

Let $\tau_0 = \gamma_0 = 0, \tau_1 = T_{N_1} = \sum_{i=1}^{N_1} \xi_i, \gamma_1 = \tau_1 + \theta_1 = T_{N_1} + \theta_1$

$$\tau_n = T_{N_1+\dots+N_n} + \sum_{i=1}^{n-1} \theta_i, \gamma_n = \tau_n + \theta_n = T_{N_1+\dots+N_n} + \sum_{i=1}^n \theta_i, n \geq 1.$$

Let's construct the sequence of the counting processes:

$$v_0(t) \equiv v([0, t]) \equiv v(t) = \max\{n \geq 0 : T_n \leq t\}, t \in [0, \tau_1).$$

$$v_r(t) \equiv v([\gamma_r, t]) = \max\{n \geq 0 : \gamma_r + (T_{N_0+N_1+\dots+N_n+n} - T_{N_0+N_1+\dots+N_n}) \leq t\}, t \in [\gamma_r, \tau_{r+1}), r = 0, 1, 2, \dots$$

By using these notations, the desired stochastic process $X(t)$ is defined as follows:

a) $X(t) = s$ for all $t \in [\tau_n, \gamma_n)$

b) $X(t) = s + \zeta_n - (S_{N_0+N_1+\dots+N_n+v_n(t)} - S_{N_0+N_1+\dots+N_n})$ for all $t \in [\gamma_n, \tau_{n+1})$

where $n = 0, 1, 2, \dots$; $\gamma_0 = 0$; $\zeta_0 = x$; $N_0 = 0$.

In this study, the process $X(t)$ will be called "a semi-Markovian random walk with delay and Weibull distributed interference of chance".

The main purpose of this study is to investigate the asymptotic behavior of the stationary moments of the process $X(t)$, as $\lambda \rightarrow 0$. For this purpose, we first discuss the ergodicity of the process $X(t)$.

3. Preliminary discussions

Firstly, we can state the following lemma from [1].

3.1. Lemma. Assume that the initial sequence of the random vectors $\{(\xi_n, \eta_n, \theta_n, \zeta_n)\}$, $n \geq 1$, satisfies the following supplementary conditions:

- 1) $E\xi_1 < \infty$; 2) $E\theta_1 < \infty$; 3) $0 < E\eta_1 < \infty$; 4) $E(\eta_1^2) < \infty$
- 5) η_1 is non-arithmetic random variable;

Then the process $X(t)$ is ergodic.

3.2. Remark. Let's now put $\varphi_X(u) \equiv \lim_{t \rightarrow \infty} E\{\exp(iuX(t))\}$, $u \in \mathbb{R}$. Using the basic identity for the random walks (see, Feller W., [5], p.514) and 3.1 Lemma, we obtain the following 3.3.

3.3. Lemma. Assume that assumptions 3.1 are satisfied and the sequence of the random variables $\{\zeta_n\}$, $n \geq 1$, which describes the discrete interference of chance has Weibull distribution with the parameters (α, λ) , $\alpha > 1$, $\lambda > 0$. Then for $u \in \mathbb{R} \setminus \{0\}$, the characteristic function $\varphi_X(u)$ of the ergodic distribution of the process $X(t)$ can be expressed by means of the characteristics of the pair $(N(x), S_{N(x)})$ and the random variable η_1 as follows:

$$(3.1) \quad \varphi_X(u) = \frac{e^{ius}}{EN(\zeta_1) + K} \int_0^\infty x^{\alpha-1} e^{-(\lambda x)^\alpha} e^{iuz} \frac{\varphi_{S_{N(x)}}(-u) - 1}{\varphi_\eta(-u) - 1} dx + \frac{K e^{ius}}{EN(\zeta_1) + K} \int_0^\infty x^{\alpha-1} e^{-(\lambda x)^\alpha} e^{iuz} \varphi_{S_{N(x)}}(-u) dx,$$

where $EN(\zeta_1) = \alpha\lambda^\alpha \int_0^\infty x^{\alpha-1} e^{-(\lambda x)^\alpha} EN(x) dx$; $\varphi_{S_{N(x)}}(-u) = E \exp(-iu S_{N(x)})$;

$$\varphi_\eta(-u) = E \exp(-iu \eta_1); K = E\theta_1/E\xi_1.$$

4. Exact formulas for the first four moments of the ergodic distribution of the process $X(t)$

The aim of this section is to express the first four moments of the ergodic distribution of the process $X(t)$ by the characteristics of the boundary functional $S_{N(x)}$ and the random variable η_1 . For this aim, introduce the following notations:

$$m_k = E(\eta_1^k), M_k(x) = E(S_{N(x)}^k), m_{k1} = \frac{m_k}{m_1}, M_{k1}(x) = \frac{M_k(x)}{M_1(x)}, k = \overline{1, 5}, x \geq 0;$$

$$E(\zeta_1^n M_k(\zeta_1)) = \alpha\lambda^\alpha \int_0^\infty x^{\alpha+n-1} e^{-(\lambda x)^\alpha} M_k(x) dx, n = \overline{0, 4}, e_k = E(\zeta_1^k), k = \overline{1, 4};$$

and for the shortness of the expressions we put:

$$E(X^k) \equiv \lim_{t \rightarrow \infty} E((X(t))^k), k = \overline{1, 4} \text{ and } \bar{X}(t) \equiv X(t) - s.$$

We can now state the first main result of this section as follows.

4.1. Theorem. *Let the conditions of 3.3 be satisfied and also $E|\eta_1|^5 < \infty$. Then the first four moments of the ergodic distribution of the process $\bar{X}(t)$ exist and can be expressed by means of the characteristics of the boundary functional $S_{N(X)}$ and the random variable η_1 as follows:*

$$(4.1) \quad E(\bar{X}) = \frac{1}{E(M_1(\zeta_1)) + Km_1} \left\{ E(\zeta_1 M_1(\zeta_1)) - \frac{1}{2} E(M_2(\zeta_1)) + \frac{1}{2} (m_{21} - 2Km_1) E(M_1(\zeta_1)) + Km_1 e_1 \right\};$$

$$(4.2) \quad E(\bar{X}^2) = \frac{1}{E(M_1(\zeta_1)) + Km_1} \left\{ E(\zeta_1^2 M_1(\zeta_1)) - E(\zeta_1 M_2(\zeta_1)) + \frac{1}{3} E(M_3(\zeta_1)) + m_{21} \left(E(\zeta_1 M_1(\zeta_1)) - \frac{1}{2} E(M_2(\zeta_1)) \right) + Km_1 (E(M_2(\zeta_1)) - 2E(\zeta_1(M_1(\zeta_1))) + A_1 E(M_1(\zeta_1)) + Km_1 e_2) \right\};$$

$$(4.3) \quad E(\bar{X}^3) = \frac{1}{E(M_1(\zeta_1)) + Km_1} \left\{ E(\zeta_1^3 M_1(\zeta_1)) - \frac{3}{2} E(\zeta_1^2 M_2(\zeta_1)) + E(\zeta_1 M_3(\zeta_1)) - \frac{1}{4} E(M_4(\zeta_1)) + \frac{3}{2} (m_{21} - 2Km_1) \left(E(\zeta_1^2 M_1(\zeta_1)) - E(\zeta_1 M_2(\zeta_1)) \right) + \frac{1}{2} (m_{21} - 2Km_1) E(M_3(\zeta_1)) + 3A_1 E(\zeta_1 M_1(\zeta_1)) - \frac{3}{2} A_1 E(M_2(\zeta_1)) + 3A_2 E(M_1(\zeta_1)) + Km_1 e_3 \right\};$$

$$(4.4) \quad E(\bar{X}^4) = \frac{1}{E(M_1(\zeta_1)) + Km_1} \left\{ E(\zeta_1^4 M_1(\zeta_1)) - 2E(\zeta_1^3 M_2(\zeta_1)) + 2E(\zeta_1^2 M_3(\zeta_1)) - E(\zeta_1 M_4(\zeta_1)) + (m_{21} - 2Km_1) \left(2E(\zeta_1^3 M_1(\zeta_1)) - 3E(\zeta_1^2 M_2(\zeta_1)) + 2E(\zeta_1 M_3(\zeta_1)) + \frac{1}{2} E(M_4(\zeta_1)) \right) + \frac{1}{5} E(M_5(\zeta_1)) + 6A_1 \left(E(\zeta_1^2 M_1(\zeta_1)) - E(\zeta_1 M_2(\zeta_1)) + \frac{1}{3} E(M_3(\zeta_1)) \right) + 6A_2 (2E(\zeta_1 M_1(\zeta_1)) - E(M_2(\zeta_1))) + 3A_3 E(M_1(\zeta_1)) + Km_1 e_4 \right\};$$

$$\text{where } A_1 = \frac{m_{21}}{2}; A_2 = \frac{m_{21}^2}{2} - \frac{m_{31}}{3}; A_3 = \frac{m_{41}}{12} - \frac{m_{31}m_{21}}{3} + \frac{m_{21}^3}{4};$$

$$A_4 = \frac{m_{21}^4}{4} - \frac{m_{31}m_{21}^2}{2} + \frac{m_{41}m_{21}}{6} + \frac{m_{31}^2}{9} - \frac{m_{51}}{30},$$

$$e_k = E(\zeta_1^k), k = \overline{1, 4}; K = E\theta_1/E\xi_1.$$

Proof. Note that the conditions of 4.1 provide the existence and finiteness of first five moments of $S_{N(x)}$ (see, Feller W., [5], p.514). And by using Taylor expansions for the characteristic functions of the variables η_1 and $S_{N(x)}$, the exact expressions (4.1)-(4.4) for the first four ergodic moments of the process $X(t)$ can be obtained. \square

5. Third-order asymptotic expansions for the first four moments of the ergodic distribution

In this section, we will obtain asymptotic expansions for the first four moments of the ergodic distribution of the process $X(t)$. For this aim, we will use the ladder variables of the random walk $S_n = \sum_{i=1}^n \eta_i$, $n \geq 1$, with initial state $S_0 = 0$.

$$\text{Let } \nu_1^+ = \min\{n \geq 1 : S_n > 0\}, \chi_1^+ = S_{\nu_1^+} = \sum_{i=1}^{\nu_1^+} \eta_i.$$

Note that, the random variables ν_1^+ and χ_1^+ are called the first strict ascending ladder epoch and ladder height of the random walk $\{S_n\}$, $n \geq 0$, respectively (see, Feller W., [5], p.391).

Let's give the following lemma:

5.1. Lemma. *Let $g(x)$ ($g : \mathbb{R}^+ \rightarrow \mathbb{R}$) be a bounded function and $\lim_{x \rightarrow \infty} g(x) = 0$. Then for any $\alpha > 1$ the following asymptotic relation holds:*

$$\lim_{\lambda \rightarrow 0} \int_0^\infty e^{-t} g\left(\frac{t^\alpha}{\lambda}\right) dt = 0.$$

Proof. Under the conditions of 5.1, for any $\varepsilon > 0$, $m(\varepsilon) > 0$ exists such that for any $x \geq m(\varepsilon)$, the inequality $|g(x)| < \varepsilon$ holds. Choose $b > 0$, such that $\int_0^b e^{-t} dt < \varepsilon$. The function $g(x)$ is bounded. Therefore, for any $\lambda < \frac{b^\alpha}{m(\varepsilon)}$, we have:

$$\begin{aligned} \left| \int_0^\infty e^{-t} g\left(\frac{t^\alpha}{\lambda}\right) dt \right| &\leq \int_0^b e^{-t} \left| g\left(\frac{t^\alpha}{\lambda}\right) \right| dt + \int_b^\infty e^{-t} \left| g\left(\frac{t^\alpha}{\lambda}\right) \right| dt \leq \\ &\leq \max_{x \geq 0} |g(x)| \int_0^b e^{-t} dt + \varepsilon \int_b^\infty e^{-t} dt \leq \varepsilon M + \varepsilon \int_0^\infty e^{-t} dt = \varepsilon(M + 1), \end{aligned}$$

where $M = \max_{x \geq 0} |g(x)|$.

Since M is finite and $\varepsilon > 0$ is arbitrary positive number, the proof of the 5.1 is completed. \square

Let's give the following lemma, which proof is similar to proof of 5.1.

5.2. Lemma. *Let $g(x)$ be defined as in 5.1 and the function $R_n(x)$ be defined as $R_n(x) \equiv x^n g(x)$, $n = -1, 0, 1, 2, \dots$. Then for each $\alpha > 1$, the following asymptotic relation is true, when $\lambda \rightarrow 0$:*

$$\int_0^\infty e^{-t} R_n\left(\frac{t^\alpha}{\lambda}\right) dt = o\left(\frac{1}{\lambda^n}\right).$$

Now, we state the following auxiliary lemma, by using 5.1 in [7]:

5.3. Lemma. *Let the condition $E|\eta_1|^3 < \infty$ be satisfied. Then we can write the following asymptotic expansions, as $\lambda \rightarrow 0$:*

$$(5.1) \quad E(M_1(\zeta_1)) = E(\zeta_1) + \frac{1}{2}\mu_{21} + o(\lambda),$$

$$(5.2) \quad E(M_2(\zeta_1)) = E(\zeta_1^2) + \mu_{21}E(\zeta_1) + \frac{1}{3}\mu_{31} + o(1),$$

$$(5.3) \quad E(M_3(\zeta_1)) = E(\zeta_1^3) + \frac{3}{2}\mu_{21}E(\zeta_1^2) + \mu_{31}E(\zeta_1) + o\left(\frac{1}{\lambda}\right),$$

$$(5.4) \quad E(M_4(\zeta_1)) = E(\zeta_1^4) + 2\mu_{21}E(\zeta_1^3) + 2\mu_{31}E(\zeta_1^2) + o\left(\frac{1}{\lambda^2}\right),$$

$$(5.5) \quad E(M_5(\zeta_1)) = E(\zeta_1^5) + \frac{5}{2}\mu_{21}E(\zeta_1^4) + \frac{10}{3}\mu_{31}E(\zeta_1^3) + o\left(\frac{1}{\lambda^3}\right).$$

Proof. Using 5.2 in this paper and 5.1 in [7], we can obtained the asymptotic expansions (5.1)-(5.5), as $\lambda \rightarrow 0$. \square

Now, we can state the first main result of this section as follows:

5.4. Theorem. *Let the conditions of 4.1 be satisfied. Then the following asymptotic expansion can be written for the first four moments of the ergodic distribution of the*

process $\bar{X}(t)$, as $\lambda \rightarrow 0$:

$$(5.6) \quad E(\bar{X}) = C_{21}(\alpha) \frac{1}{\lambda} + B_{11}(\alpha) + B_{12}(\alpha)\lambda + o(\lambda),$$

$$(5.7) \quad E(\bar{X}^2) = C_{31}(\alpha) \frac{1}{\lambda^2} + B_{21}(\alpha) \frac{1}{\lambda} + B_{22}(\alpha) + o(1),$$

$$(5.8) \quad E(\bar{X}^3) = C_{41}(\alpha) \frac{1}{\lambda^3} + B_{31}(\alpha) \frac{1}{\lambda^2} + B_{32}(\alpha) \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right),$$

$$(5.9) \quad E(\bar{X}^4) = C_{51}(\alpha) \frac{1}{\lambda^4} + B_{41}(\alpha) \frac{1}{\lambda^3} + B_{42}(\alpha) \frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right),$$

where $C_k(\alpha) = \Gamma(1 + k/\alpha)$; $C_{k1}(\alpha) = \frac{C_k(\alpha)}{kC_1(\alpha)}$, $k = \overline{1, 5}$, $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$,

$$B_{11}(\alpha) = \frac{1}{2} \left[m_{21} - \frac{C_{21}(\alpha)}{C_1(\alpha)} (\mu_{21} + 2Km_1) \right],$$

$$B_{12}(\alpha) = \frac{C_{21}(\alpha)}{4C_1^2(\alpha)} (\mu_{21}^2 + 4\mu_{21}Km_1 + 4K^2m_1^2) - \frac{1}{6C_1(\alpha)} (\mu_{31} + 3\mu_{21}m_1K + 3m_2K),$$

$$B_{21}(\alpha) = \frac{1}{2} \left[2C_{21}(\alpha)m_{21} - \frac{C_{31}(\alpha)}{C_1(\alpha)} (\mu_{21} + 2Km_1) \right],$$

$$B_{22}(\alpha) = \frac{C_{31}(\alpha)}{4C_1^2(\alpha)} (\mu_{21}^2 + 4K^2m_1^2 + 4\mu_{21}Km_1) - \frac{C_{21}(\alpha)}{2C_1(\alpha)} (m_{21}\mu_{21} - 2Km_2) + \frac{3m_{21}^2 - 2m_{31}}{6},$$

$$B_{31}(\alpha) = \frac{3C_{31}(\alpha)}{2} m_{21} - \frac{C_{41}(\alpha)}{2C_1(\alpha)} (\mu_{21} + 2Km_1),$$

$$B_{32}(\alpha) = \frac{C_{21}(\alpha)}{2} (3m_{21}^2 - 2m_{31}) + \frac{2C_{41}(\alpha)}{3C_1^2(\alpha)} (\mu_{21}^2 + \frac{3}{2}\mu_{21}Km_1 + \frac{3}{2}K^2m_1^2) - \frac{3C_{31}(\alpha)}{4C_1(\alpha)} (m_{21}\mu_{21} + 2Km_2),$$

$$B_{41}(\alpha) = 6C_{41}(\alpha) m_{21} - \frac{C_{51}(\alpha)}{2C_1(\alpha)} (\mu_{21} + 2Km_1),$$

$$B_{42}(\alpha) = C_{31}(\alpha) (3m_{21}^2 - 2m_{31} + 6m_{21}\mu_{21} - 3\mu_{31} - 12Km_1\mu_{21}) + \frac{C_{51}(\alpha)}{4C_1^2(\alpha)} (\mu_{21}^2 + 4\mu_{21}Km_1 + 4K^2m_1^2) - \frac{C_{41}(\alpha)}{C_1(\alpha)} (3m_{21}\mu_{21} - 3\mu_{21}^2 - 6Km_2 - 10Km_1\mu_{21} - 8K^2m_1^2).$$

Proof. Firstly, we obtain the asymptotic expansion for the expectation of the ergodic distribution of the process $\bar{X}(t)$, as $\lambda \rightarrow 0$. For this aim, the exact formula (4.1) was obtained for $E(\bar{X})$ in 4.1. For the shortness, we put

$$(5.10) \quad E(\bar{X}) = R(\lambda) J(\lambda),$$

where $R(\lambda) = \frac{1}{E(M_1(\zeta_1)) + Km_1}$; $J(\lambda) = J_1(\lambda) + J_2(\lambda)$;

$$J_1(\lambda) = E(\zeta_1 M_1(\zeta_1)) - \frac{1}{2} E(M_2(\zeta_1));$$

$$J_2(\lambda) = \frac{1}{2} (m_{21} - 2Km_1) E(M_1(\zeta_1)) + Km_1 e_1.$$

By using Lemma 5.3 , we get the following expansion, as $\lambda \rightarrow 0$:

$$(5.11) \quad J_1(\lambda) = \frac{\Gamma(1+2/\alpha)}{2\lambda^2} - \frac{\mu_{31}}{6} + o(1).$$

Using 5.3 , we obtain the following asymptotic expansion for $J_2(\lambda)$, as $\lambda \rightarrow 0$:

$$(5.12) \quad J_2(\lambda) = \frac{\Gamma(1+1/\alpha)}{2\lambda} m_{21} - \frac{1}{4}(m_{21} - 2Km_1)\mu_{21} + o(1).$$

By using asymptotic expansions (5.11) and (5.12), we get:

$$(5.13) \quad J(\lambda) = \frac{\Gamma(1+2/\alpha)}{\lambda^2} \left[\frac{1}{2} + \frac{m_{21}\Gamma(1+1/\alpha)}{\Gamma(1+2/\alpha)} \lambda + \frac{1}{\Gamma(1+2/\alpha)} \left(\frac{1}{4}(m_{21} - 2Km_1)\mu_{21} - \frac{1}{6}\mu_{31} \right) \lambda^2 + o(\lambda^2) \right].$$

Analogically, we calculate:

$$(5.14) \quad R(\lambda) = \frac{\lambda}{\Gamma(1+1/\alpha)} \left[1 - \frac{\mu_{21} + 2Km_1}{2\Gamma(1+1/\alpha)} \lambda + \frac{\mu_{21}^2 + 4\mu_{21}Km_1 + 4K^2m_1^2}{4\Gamma^2(1+1/\alpha)} \lambda^2 + o(\lambda^2) \right].$$

Taking into account the asymptotic expansions (5.13) and (5.14), we obtain the following asymptotic expansion, as $\lambda \rightarrow 0$:

$$(5.15) \quad \begin{aligned} R(\lambda) J(\lambda) &= \frac{\Gamma(1+2/\alpha)}{2\Gamma(1+1/\alpha)} \frac{1}{\lambda} - \frac{1}{2} \left[m_{21} - (\mu_{21} + 2Km_1) \frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha)} \right] \\ &+ \left[(\mu_{21}^2 + 4\mu_{21}Km_1 + 4K^2m_1^2) \frac{\Gamma(1+2/\alpha)}{8\Gamma^3(1+1/\alpha)} \right. \\ &\left. - (\mu_{31} + 3\mu_{21}Km_1 + 3Km_2) \frac{1}{6\Gamma(1+1/\alpha)} \right] \lambda + o(\lambda). \end{aligned}$$

Substituting (5.15) in (5.10), we finally get the asymptotic expansion (5.6) for $E(\bar{X})$, as $\lambda \rightarrow 0$.

Now, we can analogically derive the asymptotic expansion for the second moment of the ergodic distribution of the process $\bar{X}(t)$. For this aim, the exact formula (4.2) was obtained for $E(\bar{X}^2)$ in 4.1. For the shortness, we put

$$(5.16) \quad E(\bar{X}^2) = R(\lambda) J'(\lambda),$$

where $J'(\lambda) = J_3(\lambda) + J_4(\lambda)$; $J_3(\lambda) = E(\zeta_1^2 M_1(\zeta_1)) - E(\zeta_1 M_2(\zeta_1)) + \frac{1}{3}E(M_3(\zeta_1))$;

$$\begin{aligned} J_4(\lambda) &= (m_{21} - 2Km_1)E(\zeta_1 M_1(\zeta_1)) - \frac{1}{2}(m_{21} - 2Km_1)E(M_2(\zeta_1)) \\ &+ A_1 E(M_1(\zeta_1)) + Km_1 e_2. \end{aligned}$$

Using 5.3 , we obtain the following asymptotic expansion for $J_3(\lambda)$, as $\lambda \rightarrow 0$:

$$(5.17) \quad J_3(\lambda) = \frac{1}{3} \frac{\Gamma(1+3/\alpha)}{\lambda^3} + o\left(\frac{1}{\lambda}\right).$$

Taking 5.3 into account, we write the following asymptotic expansion for $J_4(\lambda)$, as $\lambda \rightarrow 0$:

$$(5.18) \quad J_4(\lambda) = \frac{m_{21}}{2} \frac{\Gamma(1+2/\alpha)}{\lambda^2} + A_1 \frac{\Gamma(1+1/\alpha)}{\lambda} + \frac{A_1 \mu_{21}}{2} - \frac{(m_{21} - Km_1)}{6} \mu_{31} + o(1).$$

Using asymptotic expansions (5.17) and (5.18), we obtain the following asymptotic expansion for $J'(\lambda)$, as $\lambda \rightarrow 0$:

$$(5.19) \quad J'(\lambda) = \frac{\Gamma(1+3/\alpha)}{\lambda^3} \left[\frac{1}{3} + \frac{m_{21}}{2} \frac{\Gamma(1+2/\alpha)}{\Gamma(1+3/\alpha)} \lambda + A_1 \frac{\Gamma(1+1/\alpha)}{\Gamma(1+3/\alpha)} \lambda^2 + \left(\frac{A_1 \mu_{21}}{2} - \frac{(m_{21} - Km_1)}{6} \mu_{31} \right) \frac{1}{\Gamma(1+3/\alpha)} \alpha + o(1) \right].$$

Substituting asymptotic expansions (5.14) and (5.19) in the formula (5.16), and carrying out the corresponding calculation, we finally get the asymptotic expansion (5.7) for $E(\bar{X}^2)$, as $\lambda \rightarrow 0$.

Analogically, we can calculate the asymptotic expansions for the third and fourth moments of the ergodic distribution of the process $\bar{X}(t)$.

This completes the proof of 5.4. \square

5.5. Corollary. *Let the conditions of 5.4 are satisfied. Then the following asymptotic expansion can be written for the variance of the ergodic distribution of the process $\bar{X}(t)$, as $\lambda \rightarrow 0$:*

$$\text{Var}(\bar{X}) = [C_{31}(\alpha) - C_{21}^2(\alpha)] \frac{1}{\lambda^2} + [B_{21}(\alpha) - 2B_{11}(\alpha)C_{12}(\alpha)] \frac{1}{\lambda} + [B_{22}(\alpha) - B_{11}^2(\alpha) - 2C_{21}(\alpha)B_{12}(\alpha)] + o(1).$$

5.6. Remark. Thus, we obtained the asymptotic expansions for the first four ergodic moments of the process $\bar{X}(t)$. Using these moments, it is possible to calculate skewness (γ_3) and kurtosis (γ_4) of the ergodic distribution of $\bar{X}(t)$:

$$\gamma_3 = \frac{E(X-a)^3}{\sigma^3}, \quad \gamma_4 = \frac{E(X-a)^4}{\sigma^4} - 3 \quad \text{where } a = E(X), \quad \sigma^2 = \text{Var}(X).$$

5.7. Corollary. *Under the conditions of 5.4, the following asymptotic expansions can be written for the skewness (γ_3) and kurtosis (γ_4) of the ergodic distribution of the process $\bar{X}(t)$, as $\lambda \rightarrow 0$:*

$$\gamma_3 = \frac{C_{41}(\alpha) - 3C_{21}(\alpha)C_{31}(\alpha) + 2C_{21}^3(\alpha)}{[C_{31}(\alpha) - C_{21}^2(\alpha)]\sqrt{C_{31}(\alpha) - C_{21}^2(\alpha)}} + O(\lambda),$$

$$\gamma_4 = \frac{C_{51}(\alpha) + 6C_{31}(\alpha)C_{21}^2(\alpha) - 4C_{21}(\alpha)C_{41}(\alpha) - 3C_{21}^4(\alpha)}{[C_{31}(\alpha) - C_{21}^2(\alpha)]^2} - 3 + O(\lambda).$$

6. Conclusions

In this study, some stationary characteristics of the process $X(t)$ are investigated by using analytical and asymptotic methods, whenever the sequence of random variables $\{\zeta_n\}$, $n \geq 1$ which describes the discrete interference of chance has Weibull distribution with parameters (α, λ) , $\alpha > 1$, $\lambda > 0$. The simple forms of the asymptotic expansions allow us to observe how the initial random variables ξ_1, η_1 , and ζ_1 influence to the stationary characteristics of the process $X(t)$.

Note that it is important to obtain the similar results for other types of discrete interference of chance by using the methods introduced in this paper.

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