A GENERALIZATION OF JORDAN’S INEQUALITY AND AN APPLICATION

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Abstract
In this article, a new generalization of Jordan’s inequality
\[ \sum_{k=1}^{n} \mu_k (\theta^t - x^t)^k \leq \frac{\sin x}{x} - \frac{\sin \theta}{\theta} \leq \sum_{k=1}^{n} \omega_k (\theta^t - x^t)^k \]
for \( t \geq 2, n \in \mathbb{N} \) and \( \theta \in (0, \pi] \) is established, where the coefficients \( \mu_k \) and \( \omega_k \) are defined by recursion formulas, and are the best possible. As an application, Yang’s inequality is refined.

Keywords: Jordan’s inequality, Yang’s inequality, L’Hôpital’s rule, Refinement, Application.

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1. Introduction
The well-known Jordan’s inequality (see [2, 5], [3, p. 143], [7, p. 269] and [10, p. 33]) states that
\[ \frac{2}{\pi} \leq \frac{\sin x}{x} < 1 \]
for \( 0 < |x| \leq \frac{\pi}{2} \). Equality in (1.1) is valid if and only if \( x = \frac{\pi}{2} \).

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Jordan’s inequality and its refinements have important applications in several mathematical areas such as calculus and trigonometry, where specially the theory of limits are involved in [25]. These are important tools in approximating the Riemann zeta function $\zeta(x)$ in [8], in improving Yang’s inequality in [29] and its generalization, which play an important role in the theory of distribution of values of functions. Therefore, many mathematicians have struggled to refine, generalize and apply it. For more detailed information, please refer to [7, pp. 274–275] and [1, 4, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 24, 25, 26, 28, 30, 33, 34, 35], especially [11, 20], and related references therein.

In [1, 9, 15, 16, 17, 18, 19], among other things, Jordan’s inequality had been refined as

\[
\frac{1}{\pi} x (\pi^2 - 4x^2) \leq \sin x - \frac{2}{\pi} x \leq \frac{\pi - 2}{\pi^3} x (\pi^2 - 4x^2) \tag{1.2}
\]

In [35], a stronger sharp double inequality for $x \in (0, \frac{\pi}{2}]$ was obtained:

\[
\frac{12 - \pi^2}{10\pi^2} (\pi^2 - 4x^2)^2 \leq \sin x - \frac{2}{\pi} \leq \frac{\pi - 3}{\pi^2} (\pi^2 - 4x^2)^2 \tag{1.3}
\]

Recently, the following general refinement of Jordan’s inequality was shown in [13]:

\[
\frac{2}{\pi} + \sum_{k=1}^{n} \alpha_k (\pi^2 - 4x^2)^k \leq \sin x - \frac{2}{\pi} \leq \frac{2}{\pi} + \sum_{k=1}^{n} \beta_k (\pi^2 - 4x^2)^k \tag{1.4}
\]

where the constants

\[
\alpha_k = \frac{(-1)^k}{(4\pi)^k k!} \sum_{i=1}^{k+1} \left( \frac{2}{\pi} \right)^i c_{i-1}^k \sin \left( \frac{k+i}{2\pi} \right) \tag{1.5}
\]

and

\[
\beta_k = \begin{cases} 
\frac{1 - 2/\pi - \sum_{i=1}^{n-1} \alpha_i 2^i}{\pi 2^n}, & k = n \\
\alpha_k, & 1 \leq k < n
\end{cases}
\]

with

\[
c_i^k = \begin{cases} 
(i + k - 1)c_{i-1}^{k-1} + c_{i-1}^{k-1}, & 0 \leq i \leq k \\
1, & i = 0 \\
0, & i > k
\end{cases} \tag{1.6}
\]

are the best possible.

In [28], as a generalization of Jordan’s inequality (1.1), the following sharp inequality

\[
\frac{1}{2\tau^2} \left[ (1 + \lambda) \left( \frac{\sin \theta}{\theta} - \cos \theta \right) - \theta \sin \theta \right] \left( 1 - \frac{x^\tau}{\theta^\tau} \right)^2 \leq \frac{\sin x}{x} - \sin \theta - \frac{1}{\lambda} \left( \frac{\sin \theta}{\theta} - \cos \theta \right) \left( 1 - \frac{x^\lambda}{\theta^\lambda} \right) \leq \left[ 1 - \frac{\sin \theta}{\theta} - \frac{1}{\lambda} \left( \frac{\sin \theta}{\theta} - \cos \theta \right) \right] \left( 1 - \frac{x^\tau}{\theta^\tau} \right)^2 \tag{1.8}
\]

was obtained for $0 < x \leq \theta \in (0, \frac{\pi}{2}]$, $\tau \geq 2$ and $\lambda \leq x \leq 2\tau$. Equalities in (1.8) hold if and only if $x = \theta$. The coefficients of the term $\left( 1 - \frac{x^\tau}{\theta^\tau} \right)^2$ are the best possible. If
1 ≤ τ ≤ \frac{5}{3} and either λ ≠ 0 or λ ≥ 2τ, then inequality (1.8) is reversed. In particular, when θ = \frac{τ}{2}, inequality (1.8) becomes

\begin{equation}
\frac{4λ + 4 - \pi^2}{4π^2\pi^{2\tau+1}}(π^\tau - 2^\tau x^\tau)^2 ≤ \frac{\sin x}{x} - \frac{2}{\pi} - \frac{2}{\lambdaπ^{\lambda+1}}(π^λ - 2^λ x^λ)
\end{equation}

\begin{equation}
≤ \frac{λπ - 2\lambda - 2}{\lambdaπ^{\lambda+1}}(π^\tau - 2^\tau x^\tau)^2
\end{equation}

for 0 < x ≤ \frac{π}{2}, τ ≥ 2 and τ ≤ λ ≤ 2τ. If 1 ≤ τ ≤ \frac{5}{3} and either λ ≠ 0 or λ ≥ 2τ, then the inequality (1.9) is reversed. If we take (τ, λ) = (2, 2) in (1.9), then the inequality (1.3) can be deduced.

For recent developments of refinements, generalizations and applications of Jordan’s inequality, please refer to the survey paper [20] and related references therein.

The first aim of this paper is to generalize inequalities (1.4) and (1.8) as the following

Theorem 1.1.

1.1. Theorem. For 0 < x ≤ θ < π, n ∈ N and t ≥ 2, the inequality

\begin{equation}
\sum_{k=1}^{n} \mu_k(\theta^t - x^t)^k ≤ \frac{\sin x}{x} - \frac{\sin θ}{θ} ≤ \sum_{k=1}^{n} \omega_k(\theta^t - x^t)^k
\end{equation}

holds with equalities if and only if x = θ, where the constants

\begin{equation}
\mu_k = \left\{ \begin{array}{ll}
(-1)^k \sum_{i=1}^{k+1} \frac{a_{k+1}^{i+1}}{k!^{t^k}} \sum_{i=1}^{k+1} a_{i-1}^{k+1} \theta^{k-i-kt} \sin \left( \theta + \frac{k+i-1}{2\pi} \right), & k = n, \\
\mu_k, & 1 \leq k < n
\end{array} \right.
\end{equation}

and

\begin{equation}
\omega_k = \left\{ \begin{array}{ll}
1 - \frac{\sin θ/θ - \sum_{i=1}^{n-1} \mu_i θ^{ti}}{\theta^{tn}}, & k = n, \\
\mu_k, & 1 \leq k < n
\end{array} \right.
\end{equation}

with

\begin{equation}
a_i^k = \left\{ \begin{array}{ll}
\frac{a_{i-1}^{k-1} + [i + (k - 1)(t - 1)]a_{i-1}^{k-1}}{1}, & 0 < i \leq k, \\
1, & i = 0, \\
0, & i > k
\end{array} \right.
\end{equation}

are the best possible.

1.2. Remark. Taking t = 2 in (1.10) yields inequality (1.4). Letting n = 2 in (1.10) leads to (1.8) for λ = τ = 2.

The second aim of this paper is to apply Theorem 1.1 to refine Yang’s inequality [29] as follows.

1.3. Theorem. Let 0 ≤ λ ≤ 1, 0 < x ≤ θ < π, t ≥ 2 and A_i > 0 with \sum_{i=1}^{n} A_i ≤ π for n ∈ N. If m ∈ N and n ≥ 2, then

\begin{equation}
L_m(n, λ) ≤ H(n, λ) ≤ R_m(n, λ),
\end{equation}
where

\[(1.15) \quad L_m(n, \lambda) = \left(\frac{n}{2}\right) \lambda^2 \pi^2 \left[\sin \frac{\theta}{\theta} + \sum_{k=1}^{m} 2^{-kt} \mu_k \left(2^{t} \pi^i - \lambda^t \pi^j \right)^k\right]^2 \cos^2 \left(\frac{\lambda}{2} \pi\right),\]

\[(1.16) \quad H(n, \lambda) = (n - 1) \sum_{k=1}^{n} \cos^2 (\lambda A_k) - 2 \cos(\lambda \pi) \sum_{1 \leq i < j \leq n} \cos(\lambda A_i) \cos(\lambda A_j),\]

\[(1.17) \quad R_m(n, \lambda) = \left(\frac{n}{2}\right) \lambda^2 \pi^2 \left[\sin \frac{\theta}{\theta} + \sum_{k=1}^{m} 2^{-kt} \omega_k \left(2^{t} \pi^i - \lambda^t \pi^j \right)^k\right]^2 \cos^2 \left(\frac{\lambda}{2} \pi\right),\]

and \(\mu_k\) and \(\omega_k\) are defined by (1.11).

2. Lemmas

To prove our main results, the following lemmas are necessary.

2.1. Lemma. For \(x > 0\), let \(u_0(x) = \frac{\sin x}{x}\) and \(u_k(x) = \frac{u_{k-1}'(x)}{x^r}\) for \(k \in \mathbb{N}\) and \(r \geq 1\). Then

\[(2.1) \quad u_k(x) = \sum_{i=1}^{k+1} a_{i-1}^{k+1} \sin \left(x + (i + k - 1)\pi/2\right) \frac{x^{kr+i}}{x^{kr+i}},\]

where \(a_{i}^{k+1}\) is defined by (1.13).

Proof. It is apparent that

\[u_1(x) = x^{-r} \left(\frac{\sin x}{x}\right)' = x^{-1-r} \cos x - x^{-2-r} \sin x,\]

which tells us that the formula (2.1) is valid for \(k = 1\).

Now assume the formula (2.1) holds for some given \(k > 1\). Direct computation and utilization of (1.13) gives

\[u_{k+1} = \sum_{i=1}^{k+1} a_{i-1}^{k+1} \left[\frac{1}{x^{kr+i+r}} \cos \left(x + \frac{k + i - 1}{2} \pi\right)ight.\]

\[= \frac{a_{0}^{k}}{x^{kr+i+r}} \cos \left(x + \frac{k}{2} \pi\right) - \frac{(kr + k + 1)a_{1}^{k+1}}{x^{kr+r+k+2}} \sin(x + k\pi)\]

\[= \sum_{i=0}^{k} a_{i}^{k+1} \left[\frac{1}{x^{kr+i+r}} \sin \left(x + \frac{k + i}{2} \pi\right)\right.\]

\[= \frac{a_{0}^{k+1}}{x^{kr+i+r}} \sin \left(x + \frac{k + 1}{2} \pi\right) + \frac{a_{k+1}^{k+1}}{x^{kr+r+k+2}} \sin(x + (k + 1)\pi)\]

\[= \sum_{i=0}^{k-1} a_{i+1}^{k+1} \left[\frac{1}{x^{kr+i+r}} \sin \left(x + \frac{k + i + 1}{2} \pi\right)\right.\]

\[= \sum_{i=1}^{k+1} a_{i-1}^{k+1} \sin \left(x + \frac{k + i}{2} \pi\right).\]

By mathematical induction, Lemma 2.1 is proved.
2.2. Lemma. For \( x > 0 \) and \( k \in \mathbb{N} \), let

\[
v_1(x) = \sum_{i=1}^{k+1} a_i x^{k-i+1} \sin \left( x + \frac{k+i-1}{2} \pi \right)
\]

and \( v_{j+1}(x) = \frac{1}{x} v_j'(x) \) for \( j \in \mathbb{N} \). Then

\[
v_j(x) = \sum_{i=0}^{k-j+1} b_i x^{k-i-j+1} \sin \left( x + \frac{k+i+j-1}{2} \pi \right)
\]

is valid for \( j \in \mathbb{N} \), where \( b_i^1 = a_i^k \), \( b_0^1 = 1 \) and

\[
b_i^j = b_i^{j-1} - (k-i-j+3) b_i^{j-1}, \quad 0 < i \leq k-j+1, \ j > 1.
\]

Proof. When \( j = 1 \), the formula (2.2) is clearly valid.

By induction, suppose that the formula (2.2) holds for some \( j > 1 \). Since \( k-j+1 > k-(j+1)+1 \), it can be deduced from (2.3) that \( b_{k-j+1}^j = b_{k-j+1}^j - b_{k-j}^j = 0 \). Thus,

\[
v_{j+1}(x) = \frac{1}{x} \left\{ \sum_{i=0}^{k-j} b_i^j \left[ (k-i-j+1)x^{k-i-j} \sin \left( x + \frac{k+i+j-1}{2} \pi \right) \right. \right.
\]

\[
+ x^{k-i-j+1} \cos \left( x + \frac{k+i+j-1}{2} \pi \right) \left. \right] + b_{k-j}^j \cos(x+k\pi) \right\)
\]

\[
= b_{k-j}^j x^{k-j} \sin \left( x + \frac{k+j}{2} \pi \right)
\]

\[
+ \sum_{i=0}^{k-j-1} \left[ b_{i+1}^j - (k-i-j+1) b_i^j \right] x^{k-i-j+1} \sin \left( x + \frac{k+i+j+1}{2} \pi \right)
\]

\[
= b_{k-j}^j x^{k-j} \sin \left( x + \frac{k+j}{2} \pi \right)
\]

\[
+ \sum_{i=0}^{k-j-1} b_{i+1}^j x^{k-i-j+1} \sin \left( x + \frac{k+i+j+1}{2} \pi \right)
\]

\[
= \sum_{i=0}^{k-j} b_{i+1}^j x^{k-i-j} \sin \left( x + \frac{k+i+j}{2} \pi \right).
\]

By mathematical induction, the formula (2.2) is proved. \( \square \)

2.3. Lemma. [23] Let \( f \) and \( g \) be continuous on \([a, b]\) and differentiable in \((a, b)\) such that \( g'(x) \neq 0 \) in \((a, b)\). If \( \frac{f'(x)}{g'(x)} \) is increasing (or decreasing) in \((a, b)\), then the functions \( \frac{f(x) - f(b)}{g(x) - g(b)} \) and \( \frac{f(x) - f(a)}{g(x) - g(a)} \) are also increasing (or decreasing) in \((a, b)\).

2.4. Lemma. Let \( 0 < x \leq \theta < \pi \) and \( t \geq 2 \). Then the double inequality

\[
\left( \frac{1}{t} \sin \theta \frac{\sin \theta}{\theta^{1+t}} - \frac{\cos \theta}{\theta^t} \right) (\theta^t - x^t) \leq \sin x - \frac{\sin \theta}{\theta^t} \leq \left( \frac{1}{t} \sin \theta \frac{\sin \theta}{\theta^{1+t}} - \frac{\cos \theta}{\theta^t} \right) (\theta^t - x^t)
\]

holds with equalities if and only if \( x = \theta \), where the constants

\[
\left( \frac{1}{t} \sin \theta \frac{\sin \theta}{\theta^{1+t}} - \frac{\cos \theta}{\theta^t} \right) \text{ and } \left( \frac{1}{t} \sin \theta \frac{\sin \theta}{\theta^{1+t}} \right)
\]

are the best possible.
**Proof.** Let

\[ f(x) = \frac{\sin x}{x} - \frac{\sin \theta}{\theta}, \quad g(x) = \theta^t - x^t, \]

\[ f_1(x) = x \cos x - \sin x, \quad g_1(x) = -tx^{t+1}. \]

Then

\[ \frac{f(x)}{g(x)} = \frac{f(x) - f(0)}{g(x) - g(0)}, \quad \frac{f'(x)}{g'(x)} = \frac{f_1(x) - f_1(0)}{g_1(x) - g_1(0)}, \quad \frac{f_1'(x)}{g_1'(x)} = \frac{\sin x}{t(1 + t)x^t}. \]

Since \( \frac{\sin x}{x^t} \) is decreasing in \((0, \pi]\), then \( \frac{f_1'(x)}{g_1'(x)} \) is decreasing, and so, in virtue of Lemma 2.3, the function \( \frac{f'(x)}{g'(x)} \) is decreasing, and the function \( \frac{f(x)}{g(x)} \) is decreasing in \((0, \pi]\), thus,

\[ \frac{1}{t} \left( \sin \frac{\theta}{\theta} - \cos \frac{\theta}{\theta} \right) = \lim_{x \to \theta} \frac{f(x)}{g(x)} \leq \lim_{x \to \theta} \frac{f(x)}{g(x)} \leq \lim_{x \to \theta} \frac{f(x)}{g(x)} = \frac{1}{t} \left( 1 - \frac{\sin \theta}{\theta} \right) \]

and the two constants are proved to be the best possible. \( \square \)

### 3. Proofs of theorems

#### 3.1. Proof of Theorem 1.1

If \( n = 1 \), the inequality (1.10) becomes (2.4).

For \( n \geq 2 \), let \( t = r + 1 \) and

\[ \varphi(x) = \frac{\sin x}{x} - \frac{\sin \theta}{\theta} - \sum_{k=1}^{n-1} \mu_k (\theta^{r+1} - x^{r+1})^k, \quad \psi(x) = (\theta^{r+1} - x^{r+1})^n, \]

\[ \varphi_1(x) = \frac{\varphi(x)}{x}, \quad \varphi_{i+1}(x) = \frac{\varphi_i(x)}{x^r}, \quad \psi_1(x) = \frac{\psi(x)}{x^r}, \quad \psi_{i+1}(x) = \frac{\psi_i(x)}{x^r}, \]

where \( 2 \leq i \leq n \). Then for \( 1 \leq k \leq n - 2 \),

\[ \varphi_k(x) = u_k(x) - \left[-(r + 1)\right]^k k! \mu_k - \sum_{i=1}^{n-k-1} \frac{(i+k)!}{i!} \mu_{i+k} (\theta^{r+1} - x^{r+1})^i, \]

\[ \varphi_{n-1}(x) = u_{n-1}(x) - (r - 1)! \left[-(r + 1)\right]^{n-1} \mu_{n-1}, \]

and \( \varphi_n(x) = u_n(x) \), where \( u_k(x) \) for \( 1 \leq k \leq n \) is defined by (2.1).

In view of (2.1), it is deduced that

\[ \left[-(r + 1)\right]^k k! \mu_k = u_k(\theta) \]

for \( 1 \leq k \leq n - 1 \), hence \( \varphi_i(\theta) = 0 \) for \( 1 \leq i \leq n - 1 \). A simple calculation gives

\[ \psi_i(x) = \left[-(r + 1)\right]^i \prod_{\ell=0}^{i-1} (n - \ell)(\theta^{r+1} - x^{r+1})^{n-i} \]

for \( 1 \leq i \leq n \), consequently \( \psi_i(\theta) = 0 \) for \( 1 \leq i \leq n - 1 \). As a result, for \( 1 \leq i \leq n - 1 \),

\[ \frac{\varphi(x)}{\psi(x)} = \frac{\varphi(x) - \varphi(\theta)}{\psi(x) - \psi(\theta)}, \quad \frac{\varphi'(x)}{\psi'(x)} = \frac{\varphi'(x) - \varphi'(\theta)}{\psi'(x) - \psi'(\theta)}, \]

\[ \frac{\varphi_i'(x)}{\psi_i'(x)} = \frac{\varphi_{i+1}(x) - \varphi_{i+1}(\theta)}{\psi_{i+1}(x) - \psi_{i+1}(\theta)}, \quad \frac{\varphi_{i-1}'(x)}{\psi_{i-1}'(x)} = \frac{\varphi_{i-1}(x) - \varphi_{i-1}(\theta)}{\psi_{i-1}(x) - \psi_{i-1}(\theta)}. \]

Let \( h_1(x) = x^{nr+n+1} \) and \( h_{i+1}(x) = \frac{\psi_i'(x)}{\psi(x)} h_i'(x) \) for \( 1 \leq i \leq n \) and \( n \in \mathbb{N} \). Then it is easy to see that

\[ h_{i+1}(x) = \prod_{\ell=1}^{i} (nr + n - 2\ell + 3)x^{nr+n-2\ell+1} \]
for $1 \leq i \leq n$. Utilization of Lemma 2.1 and Lemma 2.2 leads to
\[
\frac{\varphi_{n-1}(x)}{\psi_{n-1}(x)} = \frac{\sum_{i=1}^{n+1} a_{i-1} x^{n-i+1} \sin \left( x + \frac{n+i-1}{2} \right)}{n![-(1+r)]^n x^{n+1}} = \frac{v_1(x)}{n![-(1+r)]^n h_1(x)},
\]
and, since $v_i(0) = h_i(0) = 0$ for $1 \leq i \leq n+1$,
\[
\frac{v_1(x)}{h_3(x)} = \frac{v_1(x) - v_1(0)}{h_1(x) - h_1(0)}, \quad \frac{v'_j(x)}{h'_j(x)} = \frac{v_{j+1}(x) - v_{j+1}(0)}{h_{j+1}(x) - h_{j+1}(0)},
\]
\[
\frac{v'_n(x)}{h'_n(x)} = \frac{v_{n+1}(x) - v_{n+1}(0)}{h_{n+1}(x) - h_{n+1}(0)} = (-1)^n \sin x \prod_{r=1}^n (nr + n - 2\ell + 3) x^{n-r-n+1}
\]
for $1 \leq j \leq n - 1$. Since $\frac{\sin x}{x^n}$ and $x^{-n(r-1)}$ are decreasing on $(0, \pi)$, the function $\frac{(-1)^n u_i(x)}{h_i(x)}$ is decreasing and $\frac{(-1)^n u_{i-1}(x)}{h_{i-1}(x)}$ is decreasing. Accordingly, from Lemma 2.3, it follows that the functions $\frac{(-1)^n u_i(x)}{h_i(x)}$ and $\frac{(-1)^n u_{i-1}(x)}{h_{i-1}(x)}$ for $2 \leq i \leq n$ are decreasing. Thus, the functions $\frac{(-1)^n u_i(x)}{h_i(x)}$ and $\frac{(-1)^n u_{i-1}(x)}{h_{i-1}(x)}$ are decreasing, and so $\frac{\varphi_{n-1}(x)}{\psi_{n-1}(x)}$ is decreasing in $(0, \pi)$.

Utilizing Lemma 2.3 again reveals that the functions $\frac{\varphi_j(x)}{\psi_j(x)}$ and $\frac{\varphi_{j-1}(x)}{\psi_{j-1}(x)}$ for $2 \leq j \leq n - 1$ are decreasing, which implies the decreasing monotonicity of $\frac{\varphi(x)}{\psi(x)}$ in $(0, \pi)$. By L’Hospital’s rule, it is easy to deduce that
\[
\lim_{x \to -\theta^+} \frac{\varphi(x)}{\psi(x)} = \lim_{x \to +\theta^-} \frac{\varphi'(x)}{\psi'(x)} = \lim_{x \to +\theta^-} \frac{\varphi'(x)}{\psi'(x)} = \frac{u_n(\theta)}{n![-(1+r)]^n} = \mu_n
\]
for $1 \leq i \leq n - 1$ and $\lim_{x \to 0^+} \frac{\varphi(x)}{\psi(x)} = \omega_n$, which implies $\mu_n \leq \frac{\varphi(x)}{\psi(x)} \leq \omega_n$, and so the constants $\mu_n$ and $\omega_n$ are the best possible.

By mathematical induction, the inequality (1.10) is proved. The proof of Theorem 1.1 is complete.

3.2. Proof of Theorem 1.3. It was proved in [31] and [32, (2.13)] that
\[
\sin^2(\lambda \pi) \leq \cos^2(\lambda A_1) + \cos^2(\lambda A_2) = 2 \cos^2(\lambda A_1) \cos(\lambda A_1) \cos(\lambda \pi) \leq H_{ij}
\]
(3.1)
\[
\leq 4 \sin^2 \left( \frac{\lambda \pi}{2} \right).
\]
Summing up (3.1) for $1 \leq i < j \leq n$ yields
\[
\left( \frac{n}{2} \right) \sin^2(\lambda \pi) \leq \sum_{1 \leq i < j \leq n} H_{ij} = H(n, \lambda) \leq 4 \left( \frac{n}{2} \right) \sin^2 \left( \frac{\lambda \pi}{2} \right).
\]
(3.2)
By virtue of inequality (1.10) in Theorem 1.1,
\[
4 \sin^2 \left( \frac{\lambda \pi}{2} \right) \leq \lambda^2 \sin^2 \left[ \frac{\sin \theta}{\theta} + \sum_{k=1}^{m} 2^{-k} \omega_k \left( 2^{\ell} - \ell \lambda \pi \right)^k \right]^2,
\]
(3.3)
\[
\sin^2(\lambda \pi) = 4 \cos^2 \left( \frac{\lambda \pi}{2} \right) \sin^2 \left( \frac{\lambda \pi}{2} \right)
\]
\[
\geq \lambda^2 \pi^2 \left[ \frac{\sin \theta}{\theta} + \sum_{k=4}^{m} 2^{-k} \mu_k \left( 2^{\ell} - \ell \lambda \pi \right)^k \right]^2 \cos^2 \left( \frac{\lambda \pi}{2} \right).
\]
(3.4)
Substituting (3.3) and (3.4) into (3.2) leads to (1.14). The proof of Theorem 1.3 is complete.
References

[5] Feng, Y.-F. Proof without words: Jordan’s inequality \( \frac{2x}{\pi} \leq \sin x \leq x, 0 \leq x \leq \frac{\pi}{2} \), Math. Mag. 69, 126, 1996.


[29] Yang, L. Zhī Fēnbù Lǐjí Xīn Yǎnjù (The Theory of Distribution of Values of Functions and Recent Researches), Kēxué Chūbān Shè (Science Press, Beijing, 1982). (Chinese)


